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E. Artin · W. Blaschke · L. Collatz · H. Hasse
L. Schmetterer · E. Sperner · E. Witt

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Linking Spheres

By E. C. Zeeman

Conjecture: *It is possible to link two tame unknotted n-spheres in real Euclidean k-space whenever* $n + 2 \leq k \leq 2n + 1$.

Theorem: *The conjecture is true for the top dimension* $k = 2n + 1$, *and the bottom dimension* $k = n + 2$, *and for all dimensions in between except possibly those for which* $\pi_n(S^{k-n-1}) = 0$.

The first unsolved case is the linking of two 10-spheres in 17-space, corresponding to the vanishing of the stable 4-stem.

The linking properties in real Euclidean k-space E^k and in the k-sphere S^k are the same, so we shall use whichever is the most convenient. The examples that we shall construct of two n-spheres embedded in S^k will be *tame* in the sense of both Fox and Artin [2] and Whitehead [4], namely:

(i) There is a homeomorphism of S^k onto the boundary of a $(k + 1)$-simplex, suitably subdivided, which throws each S^n onto a subcomplex.

(ii) Each S^n is the image of $S^n \times 0$ in a homeomorphism of $S^n \times E^{k-n}$ into S^k (where 0 is the origin of E^{k-n}).

Therefore there are no local singularities of either an analytic or a combinatorial nature. The spheres are *linked* if they cannot be separated by an equator: in other words there is no homeomorphism of S^k onto itself which throws one S^n into the northern hemisphere and the other into the southern hemisphere. An S^n is *unknotted* in S^k if there is a homeomorphism of S^k onto the boundary of a $(k + 1)$-simplex which throws S^n onto the boundary of an $(n + 1)$-simplex.

It follows at once that the values of k mentioned in the conjecture are the only possible dimensions in which linking can occur. For if $k < n + 1$ we cannot even embed the two n-spheres. If $k = n + 1$ one of the spheres must lie inside or outside the other by the generalised Jordan curve theorem, and so there is an equator between them. If $k > 2n + 1$, then to see that any two polyhedral n-spheres in E^k are both unknotted and unlinked, one merely has to put one's eye in general position and glance at them.

We shall employ three different types of linking, (1) for the top dimension $k = 2n + 1$, (2) for the middle dimensions $n + 2 < k < 2n + 1$, and (3) for the bottom dimension $k = n + 2$. The following three sketches of circles linked in E^3 illustrate the three types, although of course the sketch in case (2) must needs be inaccurate, because when $n = 1$ there are no middle dimensions.

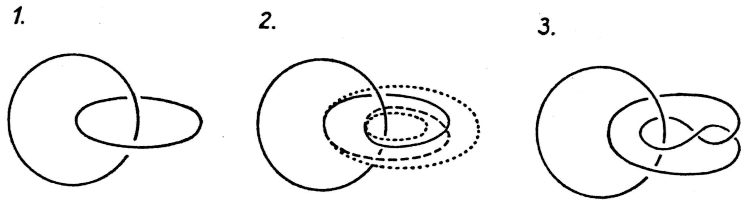

Fig. 1

Case (1) is the familiar linking of „great circles", each of which is non-homologous to zero in the complement of the other. In case (2) each sphere is homologous to zero in the complement of the other, but at least one is not homotopic to a constant in the complement of the other. Case (3) is a more delicate linkage, in which each sphere is inessential in the complement of the other, yet there is no isotopy carrying them apart; linking is achieved by Artin's method of spinning [1], and is proved by computing the fundamental group of the complement of both spheres.

Since the theorem for $n = 1$ is well known, we shall assume in the proofs that $n \geq 2$.

Proof of Cases (1) and (2)

Let $k = n + q + 1$, $2 \leq q \leq n$, and suppose that $\pi_n(S^q) \neq 0$. Represent S^k as the join of S^n_1 with S^q, and let $S^n \times S^q$ be the torus halfway between them. We have labelled S^n_1 with the subscript 1 because this will be the first of the two n-spheres to be linked. Let $f : S^n \to S^q$ be an essential (simplicial) map, and let S^n_2 be the graph of f in $S^n \times S^q$. Then

$$S^n_2 \subset S^n \times S^q \subset S^k$$

is the second of the two spheres to be linked. Clearly S^n_1 and S^n_2 are disjoint, and S^n_2 is essential in the complement $S^k - S^n_1 = E^{n+1} \times S^q$ of S^n_1. Therefore the two spheres are linked, for were they unlinked then S^n_2 would be inessential in the complement of S^n_1 (by shrinking to the south pole).

The situation is tame in the first sense because if we triangulate the join suitably so that f is simplicial, then both S^n_1 and S^n_2 appear as subpolyhedra. S^n_1 is tame in the second sense because $S^k - S^q = S^n \times E^{q+1}$ provides the necessary tubular neighbourhood. Moreover S^n_2 is a cross-section of the product bundle $S^n \times E^{q+1} \to S^n$, and so, rechoosing the origin of each fibre E^{q+1} on S^n_2, the same subset $S^k - S^q$ suffices for a tubular neighbourhood of S^n_2. Therefore the situation is tame in both senses. Also there is a linear isotopy from the cross-section S^n_2 to the cross-section S^n_1, which shows that S^n_2, like S^n_1, is unknotted in S^k.

In case (1), when $q = n$, we can either choose f to be the identity, or, more simply, choose $S^n_2 = S^q$. The two choices are isotopic.

Proof[1]) of case (3)

Let $\bar{E}^3$ be half of E^3, closed and bounded by E^2. In $\bar{E}^3$ consider the two curves C_1, C_2 linked as shown in Figure 2, with their end points in E^2.

Spin $\bar{E}^3$ about E^2 by S^{n-1}, $(n \geq 2)$. More precisely, in $\bar{E}^3 \times S^{n-1}$ identify the points $(x, y) = (x', y')$ if and only if $x = x'$ and $y = y'$ or $x = x' \in E^2$. Then the spun situation gives two n-spheres S^n_1, S^n_2 embedded in E^{n+2}. By making the curves tame, that is polygonal surrounded by tubular neighbourhoods, we ensure that the spun spheres are also tame. Clearly each curve is unknotted (if the other is removed), and so each S^n is unknotted in E^{n+2}. Therefore if the two spheres were unlinked $\pi_1(E^{n+2} - S^n_1 - S^n_2)$ would be free on two generators. By an immediate generalisation of Artin's proof in [1],

$$\pi_1(E^{n+2} - S^n_1 - S^n_2) \cong \pi_1(\bar{E}^3 - C_1 - C_2).$$

Therefore the proof of linkage, and the proof of the theorem, is completed by:

Lemma. $\pi_1(\bar{E}^3 - C_1 - C_2)$ is not free.

Proof[2]): A moment's thought will convince the reader that the customary method in [3] of using segments and cross-overs to give generators and rela-

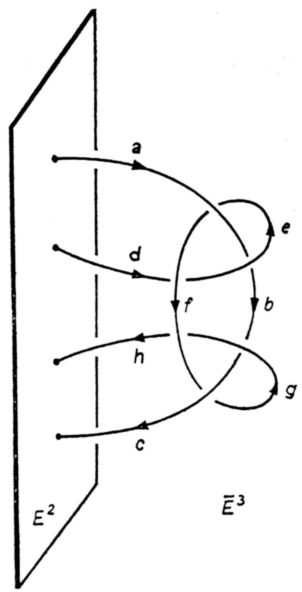

Fig. 2

[1]) *Added in proof:* M. L. Curtis has pointed out that Case (3), $n = 2$, was discovered by E. R. van Kampen, and published in this journal, VI (1928), p. 216.

[2]) I am indebted to J. A. Green for the elegant completion of this proof.

tions for $\pi = \pi_1(\bar{E}^3 - C_1 - C_2)$ is still valid, although we are dealing with curves in the half space $\bar{E}^3$ instead of loops in the whole space. Therefore we have generators $a, b, \ldots, h$ corresponding to the oriented segments shown, and relations

1. $be = ea$
2. $bg = ga$
3. $fc = cg$
4. $fa = ae$
5. $df = fe$
6. $hf = fg$

Let $x = c^{-1}a$ replace the generator a. Relations 5, 6 render the generators d, h redundant. Relations 3, 4, 5 may be used to express b, f, e in terms of c, g, x:

2. $b = gcg^{-1}$
3. $f = cgc^{-1}$
4. $e = a^{-1}fa = x^{-1}c^{-1}fcx = x^{-1}gx.$

The remaining relation 1 becomes

1. $gcg^{-1}x^{-1}gx = x^{-1}gxcx.$

Writing the commutator $g^{-1}x^{-1}gx$ as $[g, x]$, the relation becomes

$$c[g, x] = [g, x]cx,$$

or

$$[c, [g, x]] = x.$$

We therefore have a presentation of π as

$$\pi = \{c, g, x;\ [c, [g, x]] = x\}.$$

From the relation, x is preserved in the lower descending central series. Suppose now that π is free. Then the intersection of the terms of the lower descending central series is the unit element, and so $x = 1$. Therefore the relation $[c, [g, x]] = x$ implies the relation $x = 1$. But this is not so, as is shown by the model the permutation group on three elements, taking x of order 3, and $c = g$ of order 2. Therefore π is not free.

Remark

Artin mentions in [1] that it is not in general possible to link two unknotted spheres by spinning, which is precisely what we have done. What he is referring to is that we cannot complete the curves in $\bar{E}^3$ to give loops in E^3 which have a group isomorphic to π. In fact π seems to be „too nearly free" to be realised as the group of a link in E^3.

Knots

Less is known about the analogous knot problem. In [1] Artin spun a tame knotted S^2 in E^4, and his proof generalises to S^n in E^{n+2}. At the top end of the scale, all knots of S^n come undone in E^{2n+2}, but between these dimensions it is not known[3]) whether knots can exist or not.

References

[1] E. Artin, Zur Isotopie zweidimensionaler Flächen im R_4, Abhandlungen aus dem Math. Seminar, Hamburg IV (1926), p. 174—177.

[2] R. H. Fox and E. Artin, Some wild cells and spheres in three-dimensional space, Annals of Math. 49 (1948), p. 979-990.

[3] H. Seifert and W. Threlfall, Old and new results on knots, Canadian Journal of Math. II (1950), p. 1—15.

[4] J. H. C. Whitehead, On finite cocycles and the sphere theorem, Colloquium Mathematicum VI (1958), p. 271—281.

Gonville and Caius College, Cambridge.

[3]) *Added in proof:* The author has subsequently unknotted S^n in E^k, provided $k > \frac{3}{2}(n+1)$.

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