THE GENERALISED POINCARÉ CONJECTURE

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THEOREM. *If a combinatorial n-manifold has the homotopy-type of an n-sphere then it is homeomorphic to an n-sphere, provided $n \geq 5$.*

The above theorem was proved for $n \geq 7$ by Stallings [2]. His proof can be adapted to cover the cases $n = 5, 6$ by means of the following lemma (the proof of which is given in [3]).

LEMMA. *Suppose M^n is a q-connected combinatorial n-manifold, where $q \leq n - 3$. Suppose A^q is a q-subcomplex, and B a collapsible subcomplex, both contained in the interior of M^n. Then there exists a collapsible subcomplex C in the interior of a suitable subdivision σM^n of M^n, such that $C \supset \sigma(A^q + B)$ and $\dim(C - \sigma B) \leq q + 1$.*

The lemma is useful in a variety of contexts. For the application that we need here, choose A^q to be the q-skeleton of M^n and B to be a point; then a regular neighbourhood of C is an n-ball containing A^q. Therefore if there are complementary skeletons of M^n with co-dimension at least 3, we can embed them in balls, and so, by expanding one of the balls, cover M^n by two balls. The theorem follows as in [2, Lemma 3]. Complementary skeletons of codimension at least 3 exist if and only if $n \geq 5$.

In dimensions $n = 3, 4$ there is not quite enough elbow room for the proof to work, and so these two dimensions are the only outstanding cases for which the combinatorial form of Poincaré's conjecture remains open.

The combinatorial theorem above implies the analogous differential theorem of Smale [1], because differentiable manifolds can be triangulated, but not conversely.

BIBLIOGRAPHY

1. S. Smale, *The generalised Poincaré conjecture in higher dimensions*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 373–375.

2. J. R. Stallings, *Polyhedral homotopy-spheres*, Bull. Amer. Math. Soc. vol. 66 (1960) pp. 485–488.

3. E. C. Zeeman, *Isotopies of manifolds*, to appear.

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