DIHOMOLOGY
II. THE SPECTRAL THEORIES OF A MAP

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1. Introduction

JUST as we can construct various homology theories for a topological space, so we can construct various spectral theories for a continuous map. In this paper we use dihomology to define the simplicial and Čech theories, and to prove that certain theories are isomorphic under suitable conditions; in particular, all theories agree on a simplicial map. The notions of dihomology upon which this paper depends are introduced in (8) §§ 1 to 5. The following table describes the different spectral theories that we shall be concerned with.

Homology theory	Analogous spectral theory	Method of construction
Singular { homology, cohomology	spectral sequence, spectral ring }	singular cubes (Serre (4))
Alexander cohomology ≅ Čech cohomology }	spectral ring	{ sheaves (Leray (3)), dihomology
Vietoris homology ≅ Čech homology }	semi-spectral sequence	dihomology
Simplicial { homology, cohomology (on simplicial complexes)	spectral sequence, spectral ring } (on simplicial maps)	dihomology (Zeeman (5))

By *spectral sequence* and *spectral ring* we mean canonical spectral sequence for homology and canonical spectral ring for cohomology, respectively (as defined in ((8) § 5)). The term *semi-spectral* as applied to the Čech sequence is merely a description of the structure possessed by the limit of an inverse system of spectral sequences (see ((8) § 6)).

The paper is divided into seven sections:

1. Introduction.
2. Čech theory.
3. Polyhomology.
4. Leray theory.
5. Simplicial theory.
6. Computation theorem.
7. Singular theory.

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Section 2 is concerned with the definition of the simplicial and Čech spectral theories.

Section 3 is purely algebraic, and introduces multiple facing relations. The typical problem is to prove an isomorphism between two given spectral sequences arising from two double complexes. The way we tackle it is to construct a parent spectral sequence arising from a quadruple complex, and then establish sufficient conditions (in Lemmas 5 to 8) for the parent to map down isomorphically onto each of the given spectral sequences. This technique is then applied throughout the remaining four sections.

In § 4 we prove the canonical isomorphism between the Čech and the Leray spectral rings.

Section 5 establishes, in the case of a simplicial map, an isomorphism between the Čech and simplicial theories, provided we calculate the simplicial theory on the second derived map. As a corollary, the Čech sequence is promoted in this case from being semi-spectral to spectral. The isomorphism can be regarded both as a topological invariance theorem for the simplicial theory, and as a computation theorem for the Čech theory. However, one of the main applications of the spectral sequence is to fibre bundles, where, although the spaces concerned are often polyhedra, it is seldom that the projection is given as a simplicial map. So it is worth proving a better computation theorem.

This we do in § 6, and, without much extra effort, the proof goes through for a larger class of maps called *polyfibre* maps. A polyfibre map is a generalization of a fibre map with polyhedral fibre and base, the main difference being that various singular fibres are allowed above subpolyhedra of the base. The definition includes ramified coverings, and maps of manifolds onto orbit spaces of groups acting on them. The computation theorem (Theorem 4) states that, given a polyfibre map, we can construct finite coverings of the spaces concerned, from which the Čech spectral sequence and ring can be calculated in a finite number of steps (not that many folk would ever have the energy to carry out such a calculation). However, a second part of the theorem gives a method of calculating the E^2 term only from a cell decomposition of the base and the homology of the fibres, and this is a practical proposition.

In § 7 we define the singular spectral theory for a continuous map; this is a mild generalization of Serre's definition for a fibre map. The main result is to establish (Theorem 5) a canonical homomorphism Υ^* from the Čech spectral ring to the singular spectral ring—or in other words from Leray's spectral ring to Serre's. The homomorphism Υ^* generalizes the canonical homomorphism from Čech cohomology to singular cohomology

((8) Corollary 2.7), in the sense that if the map concerned is the identity map on a space, then the spectral rings of the map collapse to the cohomology rings of the space. Finally we show that Υ^* reduces to an isomorphism on certain fibre maps (Theorem 6), and on simplicial maps (Theorem 7).

We run into a certain notational impasse, due to conflicting conventions. Throughout we are concerned with a map $f: X \to Y$, and we associate with X a complex $K = \Sigma K_p$ and with Y a complex $L = \Sigma L_q$, from which we obtain a spectral sequence $E^r_{q,p}$. The adherence to alphabetical order leads to having q as the filtering degree associated with Y, which is the base when f is a fibre map, while p turns out to be associated with the fibre. The usual notation is the other way round. However, as the letters p and q are not used very much, the change will be comparatively unobtrusive. It is most apparent in § 7, where the link-up with Serre's theory takes place. As a result of interchanging p and q, it is more natural to have degenerate singular cubes degenerate at the front rather than at the back.

2. Čech theory

Facing relations

The essence of dihomology lies in the idea of a facing relation ((8) § 2). Recall that a facing relation $\mathfrak{F}$ between two geometric chain complexes $K = \Sigma K_p$ and $L = \Sigma L_q$ is a set $\{\sigma \otimes \tau\}$ of cells of $K \otimes L$ satisfying the facing condition

$$\sigma \otimes \tau \in \mathfrak{F} \quad \text{and} \quad \sigma \otimes \tau \succ \sigma' \otimes \tau' \quad \text{implies} \quad \sigma' \otimes \tau' \in \mathfrak{F}.$$

A facing relation generates (and is logically equivalent to) a subcomplex D of the double complex $K \otimes L$. Filtering with respect to p or q, we obtain two spectral sequences. If the facing relation is right acyclic the p-filtration sequence collapses ((8) Theorem 1) to the isomorphism $\epsilon_L : H_*(D) \xrightarrow{\cong} H_*(K)$ induced by the augmentation of L. Meanwhile the q-filtration yields the (we hope) interesting spectral sequence

$$H_q(L; H_p(\mathfrak{F})) \underset{q}{\Longrightarrow} H_n(K),$$

where $H_*(\mathfrak{F})$ is the covariant stack on L of left facet homology ((8) Theorem 3).

Notation: the use of the word 'etc.'

Let us denote by $E(D)$ the above spectral sequence obtained from D by q-filtration—that is to say the sequence $E_2, E_3, \ldots, E_\infty$ of bigraded differential groups, together with $H_*(D)$ and the several interrelations. If G is an arbitrary coefficient group, let $E(D; G)$ denote the spectral sequence arising from the filtered graded group $D \otimes G$; and if R is a coefficient ring and if the complexes are multiplicative (as they all are in this paper), let $E^*(D; R)$ denote the spectral ring arising from the filtered graded ring $D \pitchfork R$. To avoid repetition we shall denote these three constructions by $E(D)$ *etc.*

More generally, suppose that D is a functor from a category $\mathfrak{X}$ to the category of double complexes, which assigns to each object $X \in \mathfrak{X}$ the double complex DX, and to each map a chain map. If we form the spectral sequence $E(DX)$ we obtain a functor $E : \mathfrak{X} \to \mathfrak{E}$ from $\mathfrak{X}$ to the category $\mathfrak{E}$ of spectral sequences. Similarly $E(DX; G)$ gives a covariant functor of two variables, the first being X, and the second being the coefficient group G, into the category of spectral sequences; and $E^*(DX; R)$ gives a functor of two variables, contravariant in X and covariant in R, into the category of spectral rings (for the definitions of the categories of spectral sequences and rings see ((**8**) § 5)). We denote all three functors by E *etc.*

It is also useful to extend the technical use of the word *etc.* to cover the following situations. Suppose that S is any statement about the spectral sequence $E(D)$. We write S *etc.* if similar statements are true for $E(D; G)$ and $E^*(D; R)$. For example, suppose the homomorphism $\pi : D \to D'$ induces isomorphisms $E(D) \xrightarrow{\cong} E(D')$, $E(D; G) \xrightarrow{\cong} E(D'; G)$, and $E^*(D'; R) \xrightarrow{\cong} E^*(D; R)$. We abbreviate this to: π induces an isomorphism $E(D) \xrightarrow{\cong} E(D')$, *etc.* Note that the second two isomorphisms are not in general implied by the first (see (**9**)), but they are if the first results from the existence of a homotopy operator (see Lemma 2). If the word *etc.* is included in the statement of a lemma or theorem, we shall omit the proof of the *etc.* if it is substantially the same as the given proof.

The simplicial spectral sequence of a simplicial map, etc.

Let $f : K \to L$ be a simplicial map between finite oriented simplicial complexes. Let $\mathfrak{F}$ be the facing relation between K and L given by

$$\mathfrak{F} = \{\sigma \otimes \tau ; f(\operatorname{st} \sigma) \cap \operatorname{st} \tau \neq \varnothing\}.$$

Alternatively we may write the condition as: $\sigma \otimes \tau \in \mathfrak{F}$ if and only if there exists some $\rho \in K$ such that $\rho \succ \sigma$ and $f\rho \succ \tau$. Clearly $\mathfrak{F}$ satisfies the facing condition, for if $\sigma \otimes \tau \in \mathfrak{F}$ and $\sigma \otimes \tau \succ \sigma' \otimes \tau'$, then the same ρ will ensure that $\sigma' \otimes \tau' \in \mathfrak{F}$. The subcomplex D generated by $\mathfrak{F}$ we may think of intuitively as a 'closed neighbourhood' of the graph of f in $|K| \times |L|$. $\mathfrak{F}$ is right acyclic because the right facet $\mathfrak{F}\sigma = \overline{f(\operatorname{st} \sigma)}$ is a cone with vertex any vertex of $f\sigma$. However $\mathfrak{F}$ is not in general left acyclic, as the example below shows. Consequently we obtain a spectral sequence,

$$H_q(L; H_p(\mathfrak{F})) \underset{q}{\Longrightarrow} H_n(K),$$

which is in general non-trivial. We define this to be the *spectral sequence* $E(f)$ *of the simplicial map* f, *etc.*

The term '*etc.*' here means the spectral sequence $E(f; G)$ over an arbitrary coefficient group, and the spectral ring $E^*(f; R)$ over an arbitrary coefficient ring. Note that in order to define the spectral ring it is necessary for the oriented simplicial complexes K, L each to be given a multiplicative structure (see ((**8**) § 4)), and as usual this is done by choosing an ordering of the vertices (see ((**2**) Chapter 4)); it is not necessary that the ordering of the vertices of K be in any way related to that of L. In Lemma 4 Corollary 1 we show that the ring structure of the spectral ring is independent of the orderings chosen. In § 5 we discuss the topological invariance of $E(f)$, *etc.*

Example. Suppose we have the very special case of a simplicial map $f: K \to L$ whose underlying continuous map $f: X \to Y$ is the projection of a fibre bundle, with base Y and fibre F. Suppose further that the covering of Y by the closed stars of L refines the covering of Y by canonical neighbourhoods of the bundle. Then the left facet $\mathfrak{F}\tau = \overline{f^{-1}(\mathrm{st}\,\tau)}$ triangulates the 'solid fibre' above $\overline{\mathrm{st}\,\tau}$, which is homeomorphic to $F \times \overline{\mathrm{st}\,\tau}$ and is retractible onto a fibre. Therefore $H_*(\mathfrak{F}\tau) \cong H_*(F)$, and the left facet homology stack $H_*(\mathfrak{F})$ is none other than the associated local coefficient bundle $H_*(F)$. The spectral sequence $E(f)$ turns out to be the familiar

$$H_q(Y; H_p(F)) \underset{q}{\Longrightarrow} H_n(X).$$

As mentioned in the introduction, p and q have interchanged their customary roles.

Nerves and oriented nerves

For Čech theory we have a choice of using either nerves or oriented nerves. Recall the definitions (see (**8**) Example iv). Let α be an open covering of the topological space X. A *Čech p-simplex* is an ordered set of $p+1$ sets of α (possibly with repetitions) having non-empty intersection. The *nerve* $N(\alpha)$ is the geometric chain complex generated by all Čech simplexes.

We now define an oriented nerve of α. An *oriented Čech p-simplex* is an oriented set of $p+1$ distinct sets of α having non-empty intersection. For each distinct set of sets of α having non-empty intersection we choose an orientation, and therefore obtain an oriented Čech simplex. The oriented nerve $N^0(\alpha)$ is the geometric chain complex generated by this chosen set of oriented Čech simplexes. There is a well-known chain equivalence between an oriented nerve $N^0(\alpha)$ and the nerve $N(\alpha)$ (see Lemma 4 and ((**2**) Theorem 3.5.4)).

In the definition below of the Čech spectral sequence $\check{E}$ of a map we use nerves. Had we started with oriented nerves, then with the same definition, using exactly the same formula for the facing relation, we would have obtained a spectral sequence, $\check{E}^0$ say. In Lemma 4 Corollary 2 at the end of this section we show that $\check{E}^0 \cong \check{E}$. Therefore to define the Čech spectral sequence of a map it does not matter whether we use nerves or oriented nerves. However, there are reasons for choosing one or the other in different contexts. For the theoretical development we use nerves in order that the theory be functorial, and in particular that the ring structure of the spectral ring be functorial.

For comparison with the simplicial theory (in § 5) we use oriented nerves. For if $f: K \to L$ is a simplicial map between finite oriented simplicial complexes, and if α, β are the star coverings of K, L, respectively, then we can choose oriented nerves to be identical with K, L, and the Čech facing relation below reduces to the simplicial facing relation above. Hence the Čech facing relation is a generalization of the latter. Also for computation (in § 6) when we are given finite coverings, it is desirable to use oriented nerves in order that the computations remain finite.

The Čech facing relation of a continuous map

Let $f: X \to Y$ be a continuous map between two topological spaces. Let α, β be (open) coverings of X, Y with nerves $K = N(\alpha)$, $L = N(\beta)$, respectively, such that α refines $f^{-1}\beta$. Let $\mathfrak{F}$ be the facing relation between K and L given by:

$$\mathfrak{F} = \{\sigma \otimes \tau ; f(\sup \sigma) \cap \sup \tau \neq \emptyset\}.$$

The facing condition is satisfied because supports expand when passing to faces.

Next we show that $\mathfrak{F}$ is right acyclic. Consider the right facet $\mathfrak{F}\sigma$. Let a be a vertex of σ; then $\sup a$ is the corresponding set of α. Since α refines $f^{-1}\beta$, there is a vertex $b \in L$ such that $\sup a \subset f^{-1}(\sup b)$. Therefore $f(\sup \sigma) \subset f(\sup a) \subset \sup b$. If $\tau \in \mathfrak{F}\sigma$, then

$$\begin{aligned} f(\sup \sigma) \cap \sup b\tau &= f(\sup \sigma) \cap \sup b \cap \sup \tau \\ &= f(\sup \sigma) \cap \sup \tau \\ &\neq \emptyset, \end{aligned}$$

so that $b\tau \in \mathfrak{F}\sigma$. Therefore $\mathfrak{F}\sigma$ is a total cone with vertex b, and so is acyclic. Consequently $\mathfrak{F}$ is right acyclic. We may summarize this little piece of argument by saying $\mathfrak{F}$ is *right acyclic by Čech cones*, because it is characteristic of Čech dihomology, and is the prototype of arguments that we shall meet with often.

Filtering the resulting double complex with respect to q, we obtain a spectral sequence which we denote by $\check{E}(f, \alpha, \beta)$, *etc.* The '*etc.*' here means the spectral sequence $\check{E}(f, \alpha, \beta; G)$ over an arbitrary coefficient group, and the spectral ring $\check{E}^*(f, \alpha, \beta; R)$ over an arbitrary coefficient ring. Note that there is no choice of multiplicative structure involved in the definition of the spectral ring, because nerves are naturally multiplicative (see ((8) § 4)). Moreover the facing relation, and therefore the functor $\check{E}$ *etc.*, are functorial in the sense of ((8) § 4), as we shall now explain.

The category of continuous maps

Let $\mathfrak{M}$ be the category of continuous maps, or in other words the *derived* category of the category $\mathfrak{A}$ of topological spaces and continuous maps. An *object* of $\mathfrak{M}$ is a continuous map f between two topological spaces. A *map* of $\mathfrak{M}$ from f to f' is a pair (ϕ, ψ) of continuous maps such that $\psi f = f'\phi$.

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ \downarrow f & & \downarrow f' \\ Y & \xrightarrow{\psi} & Y' \end{array}.$$

We now introduce a larger category $\mathfrak{M}_{\text{cov}}$ that includes coverings, and which is related to $\mathfrak{M}$ in the same way that $\mathfrak{A}_{\text{cov}}$ was related to $\mathfrak{A}$ in (**8**). An *object* (f, α, β) of $\mathfrak{M}_{\text{cov}}$ is a continuous map $f: X \to Y$ between two topological spaces, together with coverings α, β of X, Y, respectively, such that α refines $f^{-1}\beta$. A *map* $(\phi, \psi, \phi_\alpha, \psi_\beta)$ of $\mathfrak{M}_{\text{cov}}$ from (f, α, β) to (f', α', β') consists of a pair (ϕ, ψ) of continuous maps such that $\psi f = f'\phi$, and such that α refines $\phi^{-1}\alpha'$ and β refines $\psi^{-1}\beta'$, together with a pair $(\phi_\alpha, \psi_\beta)$ of simplicial approximations to (ϕ, ψ) between the relevant nerves.

$$\begin{array}{ccc} X, \alpha & \xrightarrow{\phi, \phi_\alpha} & X', \alpha' \\ \downarrow f & & \downarrow f' \\ Y, \beta & \xrightarrow{\psi, \psi_\beta} & Y', \beta' \end{array}.$$

LEMMA 1. *The Čech facing relation is functorial, and therefore induces a functor* $\check{E} : \mathfrak{M}_{\text{cov}} \to \mathfrak{E}$, *etc.*

Proof. Let $(\phi, \psi, \phi_\alpha, \psi_\beta)$ be a map of $\mathfrak{M}_{\text{cov}}$, as described above. What we have to check in order to show $\mathfrak{F}$ functorial is that $\sigma \otimes \tau \in \mathfrak{F}(f, \alpha, \beta)$ implies $\phi_\alpha \sigma \otimes \psi_\beta \tau \in \mathfrak{F}(f', \alpha', \beta')$. But this is true because ϕ_α, ψ_β are simplicial approximations of ϕ, ψ, and so

$$\begin{aligned} f'(\sup \phi_\alpha \sigma) \cap \sup \psi_\beta \tau &\supset f'\phi(\sup \sigma) \cap \psi(\sup \tau) \\ &= \psi(f(\sup \sigma) \cap \sup \tau) \\ &\neq \emptyset. \end{aligned}$$

Therefore $\phi_\alpha \otimes \psi_\beta$ maps $D(f,\alpha,\beta)$ into $D(f',\alpha',\beta')$ and induces a homomorphism $\check{E}(f,\alpha,\beta) \to \check{E}(f',\alpha',\beta')$ as required. The '*etc.*' part of the lemma follows by substituting $D \otimes G$ or $D \pitchfork R$ for D in the argument.

The idea is now to take limits. But before we can do so it is necessary to verify that the functor $\check{E}$ is independent of approximation, i.e. that the homomorphism $\check{E}(f,\alpha,\beta) \to \check{E}(f',\alpha',\beta')$ depends only on ϕ, ψ and not on ϕ_α, ψ_β. Given two choices of the latter we construct (in Lemma 3) a homotopy operator on the bigraded group D. This induces a homotopy operator on the filtered graded group, which ensures (Lemma 2) that the two choices give rise to the same homomorphism of the spectral sequences.

Homotopy operators

Let A be a filtered graded differential group, with grading ${}_nA$ and filtration A_p. Since any filtered graded group in this paper has the property that ${}_nA_p = 0$ $(p < 0)$ and ${}_nA_p = {}_nA$ $(p \geqslant n)$, we may assume that the resulting spectral sequence is convergent. A *homotopy operator* between two homomorphisms $\theta^1, \theta^2 : A \to A'$ is a homomorphism $h : A \to A'$ such that $h({}_nA) \subset {}_{n+1}A$, $h(A_p) \subset A_{p+1}$, and $\theta^1 - \theta^2 = hd + dh$.

Similarly a *homotopy operator* between two chain maps $\theta^1, \theta^2 : D \to D'$ of a double complex is a homomorphism $h : D \to D'$ such that

$$h(D_{p,q}) \subset D'_{p+1,q} + D'_{p,q+1},$$

and $\theta^1 - \theta^2 = hd + dh$. Clearly if A, A' are obtained from D, D' by filtering with respect to either p or q then a homotopy operator on D induces a homotopy operator on A.

LEMMA 2. *If there is a homotopy operator between two homomorphisms $\theta^1, \theta^2 : A \to A'$ of filtered graded groups, then $\theta^1_* = \theta^2_* : E(A) \to E(A')$, etc.*

Proof. The proof is given in ((**1**) 321, Proposition 3.1). For the '*etc.*', instead of the homotopy operator h on A use the homotopy operators $h \otimes 1, h \pitchfork 1$ on $A \otimes G, A' \pitchfork R$, respectively.

LEMMA 3. *Let $\mathfrak{F}$ be a facing relation between K, L and $\mathfrak{F}'$ a facing relation between K', L'. Suppose that $\Phi : K \to K'$, $\Psi : L \to L'$ are two acyclic carriers such that if $\sigma \otimes \tau \in \mathfrak{F}$, $\sigma' \in \Phi\sigma$, and $\tau' \in \Psi\tau$, then $\sigma' \otimes \tau' \in \mathfrak{F}'$. If $\phi^1, \phi^2 : K \to K'$ are chain maps carried by Φ, and $\psi^1, \psi^2 : L \to L'$ are chain maps carried by Ψ, then there is a homotopy operator between the chain maps $\phi^1 \otimes \psi^1$, $\phi^2 \otimes \psi^2 : D \to D'$.*

Proof. Since Φ is acyclic, ϕ^1, ϕ^2 differ by an ordinary chain homotopy $h_K : K_p \to K_{p+1}$, also carried by Φ. Similarly ψ^1, ψ^2 differ by h_L carried by Ψ. Now $h_K \otimes \psi^1$ maps D to D', for if $\sigma \otimes \tau$ is a generator of D, then $\psi^1 \tau \in \Psi\tau$, and so the hypothesis implies that $\Phi\sigma$ is contained in the left facet $\mathfrak{F}'(\psi^1\tau)$. Therefore $h_K\sigma$ is a chain of this facet, and so $(h_K\otimes\psi^1)(\sigma\otimes\tau)\in D'$. Writing ω_K for the sign-changing automorphism of K,

$$\begin{aligned}(h_K\otimes\psi^1)\,d+d(h_K\otimes\psi^1)\\ &= (h_K\otimes\psi^1)(\partial_K\otimes 1+\omega_K\otimes\partial_L)+(\partial_{K'}\otimes 1+\omega_{K'}\otimes\partial_{L'})(h_K\otimes\psi^1)\\ &= (h_K\partial_K+\partial_{K'}h_K)\otimes\psi^1+(h_K\omega_K+\omega_{K'}h_K)\otimes\partial_L\psi^1\\ &= (\phi^1-\phi^2)\otimes\psi^1\end{aligned}$$

since $h_K\omega_K+\omega_{K'}h_K = 0$. Therefore $h_K\otimes\psi^1$ is a homotopy operator between $\phi^1\otimes\psi^1$ and $\phi^2\otimes\psi^1$; similarly $\phi^2\omega_K\otimes h_L$ is a homotopy operator between $\phi^2\otimes\psi^1$ and $\phi^2\otimes\psi^2$. The sum of these two homotopy operators maps $D_{p,q}$ to $D'_{p+1,q}+D'_{p,q+1}$ and gives the result.

COROLLARY. *The functor $\check{E}$ of Lemma* 1 *is independent of approximation, etc.*

Proof. Let $(\phi,\psi,\phi_\alpha,\psi_\beta)$ be a map of $\mathfrak{M}_{\text{cov}}$. Then ϕ_α is carried by the acyclic Čech carrier $\Phi: K\to K'$ given by $\Phi\sigma = \{\sigma';\ \sup\sigma'\supset\phi(\sup\sigma)\}$, and similarly ψ_β is carried by the acyclic Čech carrier $\Psi: L\to L'$. Moreover the Čech facing relation satisfies the hypothesis of Lemma 3, for if $\sigma\otimes\tau\in\mathfrak{F}(f,\alpha,\beta)$, $\sigma'\in\Psi\sigma$, $\tau'\in\Psi\tau$, then

$$\begin{aligned}f'(\sup\sigma')\cap\sup\tau' &\supset f'\phi(\sup\sigma)\cap\psi(\sup\tau)\\ &= \psi(f(\sup\sigma)\cap\sup\tau)\\ &\neq \emptyset,\end{aligned}$$

and so $\sigma'\otimes\tau'\in\mathfrak{F}(f',\alpha',\beta')$. Applying Lemmas 2 and 3 gives the corollary.

Now the object of the exercise was to take limits. Notice that for a given continuous map f, the (f,α,β)'s form a directed set if we put $(f,\alpha,\beta)>(f,\alpha',\beta')$ whenever α refines α' and β refines β'. Therefore the $\check{E}(f,\alpha,\beta)$'s form an inverse system of spectral sequences, and we may define $\check{E}(f)$ to be the limit sequence. Since the inverse limit functor is not exact the limit sequence is only semi-spectral, unless we use a field of coefficients. In the dual case the direct limit of a system of spectral rings remains a spectral ring because the direct limit functor is exact. As in ((8) Lemma 1), where the limit of a functor on $\mathfrak{A}_{\text{cov}}$ gave a functor on $\mathfrak{A}$, so the limit of a functor on $\mathfrak{M}_{\text{cov}}$ gives a functor on $\mathfrak{M}$. Summarizing what we have proved:

THEOREM 1. *Let $f: X\to Y$ be a continuous map between two topological spaces.*

(i) *If G is a group, there is a Čech semi-spectral sequence $\check{E}(f;G)$, whose ∞-term is related to the Čech homology group $\check{H}_*(X;G)$. If G is a field, $\check{E}$ is* *spectral. $\check{E}$ is a functor, covariant on $\mathfrak{M}$ and covariant in the coefficient group G.*

(ii) *If R is a ring, there is a Čech spectral ring $\check{E}^*(f; R)$, whose ∞-term is related to the Čech cohomology ring $\check{H}^*(X; R)$. $\check{E}^*$ is a functor, contravariant on $\mathfrak{M}$ and covariant in the coefficient ring R.*

Remark 1. The Čech spectral and cohomology rings in the theorem are assumed to have been defined by cochains with arbitrary supports. There is a similar result if both are taken with compact supports, and f is a proper map.

Remark 2. Just as Čech cohomology forms the link between Alexander cohomology and the computable simplicial cohomology of polyhedra, so the above spectral ring turns out to be the link between sheaf theory and the computable simplicial theory. This is the aim of the next three sections. Finally, the Čech theory is in a convenient form for the link-up with singular theory in the last section.

We conclude this section by showing that in the construction of the Čech spectral sequences it does not matter whether we use nerves or oriented nerves.

Suppose α is a covering of X. Let $N(\alpha)$ and $N^0(\alpha)$ denote respectively the nerve and an oriented nerve of α. The chain equivalences

$$N(\alpha) \underset{\bar{\theta}}{\overset{\theta}{\rightleftarrows}} N^0(\alpha)$$

are defined as follows (see ((**2**) Chapter 3)): Let $a_0 a_1 \ldots a_p$ be a Čech simplex, where $a_0, \ldots, a_p$ are sets of α with non-empty intersection. If the α_i are not all distinct, define $\theta(a_0 a_1 \ldots a_p) = 0$. If the a_i are all distinct, let $\sigma \in N^0(\alpha)$ be the associated oriented Čech simplex, and define $\theta(a_0 a_1 \ldots a_p) = \pm \sigma$, according to whether or not the ordering $a_0, a_1, \ldots, a_p$ is in the orientation class of σ.

To define $\bar{\theta}$, choose an ordering of α. Given an oriented Čech simplex $\sigma \in N^0(\alpha)$, let $a_0, a_1, \ldots, a_p$ be the corresponding sets of α, written in the correct order. Define $\bar{\theta}\sigma = \pm a_0 a_1 \ldots a_p$, according as to whether or not the ordering is in the orientation class of σ.

Then $\theta\bar{\theta} = 1$; and $\bar{\theta}\theta$ is chain homotopic to 1, because both $\bar{\theta}\theta$ and 1 are carried by the acyclic Čech carrier $\Phi : N(\alpha) \to N(\alpha)$ given by

$$\Phi\sigma = \{\tau;\ \sup \sigma \subset \sup \tau\}.$$

Hence θ and $\bar{\theta}$ are chain equivalences. Similarly if β is a covering of Y, we have a chain equivalence $\theta : N(\beta) \to N^0(\beta)$ between the nerve and an oriented nerve of β.

Suppose we are given a map $f: X \to Y$ such that α refines $f^{-1}\beta$. The Čech facing relation

$$\mathfrak{F} = \{\sigma \otimes \tau;\, f(\sup \sigma) \cap \sup \tau \neq \emptyset\}$$

between the nerves $N(\alpha)$ and $N(\beta)$ gives rise to the spectral sequence $\check{E}(f, \alpha, \beta)$, *etc.* The same facing relation between the oriented nerves $N^0(\alpha)$ and $N^0(\beta)$ gives rise to a spectral sequence $\check{E}^0(f, \alpha, \beta)$, *etc.* Note that, whereas the spectral ring for nerves is functorial, in order to define the spectral ring for the oriented nerves it is necessary to choose orderings of α and β so as to make the oriented nerves multiplicative.

LEMMA 4. *The chain equivalences between nerves and oriented nerves induce an isomorphism* $\check{E}(f, \alpha, \beta) \xrightarrow{\cong} \check{E}^0(f, \alpha, \beta)$, *etc.*

Proof. Let $D \subset N(\alpha) \otimes N(\beta)$ and $D^0 \subset N^0(\alpha) \otimes N^0(\beta)$ denote the double complexes arising from the facing relation. Let $\theta_\alpha, \bar{\theta}_\alpha$ denote the chain equivalences between $N(\alpha)$ and $N^0(\alpha)$, and $\theta_\beta, \bar{\theta}_\beta$ those between $N(\beta)$ and $N^0(\beta)$. It is easy to verify that these induce chain maps

$$D \underset{\bar{\theta}_\alpha \otimes \bar{\theta}_\beta}{\overset{\theta_\alpha \otimes \theta_\beta}{\rightleftarrows}} D^0.$$

Then

$$(\theta_\alpha \otimes \theta_\beta)(\bar{\theta}_\alpha \otimes \bar{\theta}_\beta) = \theta_\alpha \bar{\theta}_\alpha \otimes \theta_\beta \bar{\theta}_\beta = 1.$$

On the other hand,

$$(\bar{\theta}_\alpha \otimes \bar{\theta}_\beta)(\theta_\alpha \otimes \theta_\beta) = \bar{\theta}_\alpha \theta_\alpha \otimes \bar{\theta}_\beta \theta_\beta$$

is chain homotopic to 1 by Lemma 3, since $\bar{\theta}_\alpha \theta_\alpha$, $\bar{\theta}_\beta \theta_\beta$ are carried by the acyclic Čech carriers Φ_α, Φ_β, respectively, which satisfy the hypothesis of Lemma 3, because if $\sigma \otimes \tau \in \mathfrak{F}$, $\sigma' \in \Phi_\alpha \sigma$ and $\tau' \in \Phi_\beta \tau$, then

$$f(\sup \sigma') \cap \sup \tau' \supset f(\sup \sigma) \cap \sup \tau$$
$$\neq \emptyset,$$

and so $\sigma' \otimes \tau' \in \mathfrak{F}$. Hence by Lemma 2 the induced homomorphisms between spectral sequences are isomorphisms. This completes the proof of Lemma 4.

Notice that in the '*etc.*' case of the spectral rings, the isomorphism is a ring isomorphism. This is proved by using the same orderings to define the chain maps $\bar{\theta}_\alpha, \bar{\theta}_\beta$ as were used to define the multiplicative structures on the oriented nerves $N^0(\alpha), N^0(\beta)$. As a corollary we deduce that the ring structure of $\check{E}^{0*}(f, \alpha, \beta;\, R)$ is independent of the orderings.

COROLLARY 1. *Let* $f: K \to L$ *be a simplicial map between finite oriented simplicial complexes. Then the spectral sequence* $E(f)$ *is independent of the orientations of* K, L, *etc., and the spectral ring* $E^*(f;\, R)$ *is independent of the orderings of* K, L. For we may identify K, L with oriented nerves of their star coverings α, β, and then $E(f) = \check{E}^0(f, \alpha, \beta)$, *etc.*

Return to Čech theory. Given a map $f: X \to Y$, we can take inverse limits over pairs of coverings α, β of X, Y, of both $\check{E}(f, \alpha, \beta)$ and $\check{E}^0(f, \alpha, \beta)$, and obtain the semi-spectral sequences $\check{E}(f)$ and $\check{E}^0(f)$, *etc.* The chain equivalences θ commute with simplicial maps, and so they induce isomorphisms between the limits:

COROLLARY 2. *Let f be a continuous map, and let $\check{E}(f)$ and $\check{E}^0(f)$ denote the Čech semi-spectral sequences of f defined using nerves and oriented nerves respectively. Then there is an isomorphism $\check{E}(f) \xrightarrow{\cong} \check{E}^0(f)$, etc.*

3. Polyhomology

In (8) double complexes were used to relate homology theories on single complexes. Here it is necessary to use triple, quadruple, and quintuple complexes to relate spectral theories on double complexes. We therefore generalize the notion of a facing relation to an arbitrary number of complexes.

We observe in passing that although we deal with multiple complexes we are limited in the tools for handling them. We still only have homology groups and spectral sequences at our disposal, which are respectively the single and double digestion processes. Even the triple process seems to be too complicated as yet; this is probably due to the non-distributivity of a certain lattice (6). However, for the purposes of this paper spectral sequences are quite adequate.

Multiple complexes

Let $K_{(i)}$ $(i = 1, 2, \ldots, s)$, be s geometric chain complexes. The *multiple complex*

$$K = K_{(1)} \otimes K_{(2)} \otimes \ldots \otimes K_{(s)}$$

has s gradings and s differentials

$$d_{(i)} = \omega_{(1)} \otimes \ldots \otimes \omega_{(i-1)} \otimes \partial_{(i)} \otimes 1 \otimes \ldots \otimes 1 \quad (i = 1, 2, \ldots, s),$$

where $\partial_{(i)}, \omega_{(i)}$ are respectively the boundary and the sign-changing automorphism of $K_{(i)}$. Let $\lambda, \mu, \nu, \ldots$ denote disjoint subsets of the indexing set $\{1, 2, \ldots, s\}$, and let $\lambda\mu$ denote the union of λ and μ. Let $K_\lambda = \bigotimes_{i \in \lambda} K_{(i)}$. Then $K_{\lambda\mu} = K_\lambda \tilde{\otimes} K_\mu$, where $\tilde{\otimes}$ denotes the tensor product of the factors of K_λ and K_μ suitably shuffled so that the suffixes are in the correct ordering.

We remark that the particular ordering of suffixes does not in fact matter, since two different orderings give isomorphic algebraic structures —all we ask is that one particular ordering should be adhered to throughout the discussion.

Let $\epsilon_\lambda : K_{\lambda\mu} \to K_\mu$ denote the *λ-augmentation*, or, more precisely, the epimorphism formed by tensoring the augmentations $\epsilon_{(i)}$ on $K_{(i)}$, $i \in \lambda$, with the identities on $K_{(j)}$, $j \in \mu$. We shall use the same symbol ϵ_λ to label any homomorphism induced by the λ-augmentation.

A *cell* of K is the tensor product

$$\sigma = \sigma_{(1)} \otimes \sigma_{(2)} \otimes \ldots \otimes \sigma_{(s)}$$

of cells $\sigma_{(i)} \in K_{(i)}$. We write $\sigma \succ \tau$ if each $\sigma_{(i)} \succ \tau_{(i)}$.

Multiple facing relations

A *facing relation* $\mathfrak{F}$ on K is a set of cells closed under the facing condition

$$\sigma \in \mathfrak{F} \quad \text{and} \quad \sigma \succ \tau \quad \text{implies} \quad \tau \in \mathfrak{F}.$$

Therefore $\mathfrak{F}$ generates a subcomplex D of K. Let $D_\lambda = \epsilon_\mu D$, where μ is the complement of λ. Let $\mathfrak{F}_\lambda$ be the set of cells generating D_λ. Then $\mathfrak{F}_\lambda$ is a facing relation on K_λ. The test for a cell σ_λ of K_λ to be in $\mathfrak{F}_\lambda$ is the existence of cells $\sigma_{(j)} \in K_{(j)}$, each $j \notin \lambda$, which when tensored with σ_λ give a cell of $\mathfrak{F}$ because in a geometric complex every cell $\sigma_{(j)}$ possesses at least one vertex $a_{(j)}$, and the λ-augmentation maps all the $a_{(j)}$ to 1.

Upon the various D_λ we can now form several homology groups and spectral sequences, whose interrelationship we proceed to explore. For convenience of notation we dispense with brackets.

Since $D_\lambda \subset K_\lambda$, D_λ possesses gradings $n_{(i)}$ and differentials $d_{(i)}$, $i \in \lambda$. If λ is non-empty we can define on $D_{\lambda\mu}$ the grading $n = \sum_{i \in \lambda} n_{(i)}$ and differential $d = \sum_{i \in \lambda} d_{(i)}$, and hence we can form the homology group $H_\lambda D_{\lambda\mu}$. This homology group, besides having its own grading n_λ, also has induced gradings $n_{(j)}$ and differentials $d_{(j)}$, $j \in \mu$.

If we filter $D_{\lambda\mu\nu}$ with respect to the grading n_λ, and use the differential $d_{\lambda\mu}$, we obtain the spectral sequence

$$E_{\lambda,\mu} D_{\lambda\mu\nu} : H_\lambda H_\mu D_{\lambda\mu\nu} \Rightarrow H_{\lambda\mu} D_{\lambda\mu\nu}.$$

This sequence also has induced gradings and differentials from ν. If ν is empty we abbreviate the spectral sequence to the single symbol $E_{\lambda,\mu}$.

Multiple facets

If $\sigma_\mu \in K_\mu$, the λ-facet $\mathfrak{F}_\lambda \sigma_\mu$ of σ_μ is defined by

$$\mathfrak{F}_\lambda \sigma_\mu = \{\sigma_\lambda ; \, \sigma_\lambda \tilde{\otimes} \sigma_\mu \in \mathfrak{F}_{\lambda\mu}\}.$$

The notation agrees with our previous notation of facets with respect to the facing relation $\mathfrak{F}_{\lambda\mu}$ between K_λ and K_μ. Notice that $\mathfrak{F}_\lambda \sigma_\mu$ is non-empty if and only if $\sigma_\mu \in \mathfrak{F}_\mu$. The two main properties of facets are:

(i) $\mathfrak{F}_\lambda \sigma_\mu$ is a subcomplex of K_λ. We therefore allow the same symbol $\mathfrak{F}_\lambda \sigma_\mu$ to represent also the graded differential group structure.

(ii) Taking direct sums over $\sigma_\mu \in \mathfrak{F}_\mu$, $\Sigma \mathfrak{F}_\lambda \sigma_\mu \tilde{\otimes} \sigma_\mu = D_{\lambda\mu}$.

We say that $\mathfrak{F}$ is (λ, μ)-*acyclic* if, for each $\sigma_\mu \in \mathfrak{F}_\mu$, the facet $\mathfrak{F}_\lambda \sigma_\mu$ is acyclic. If $\mathfrak{F}$ is (λ, μ)-acyclic then μ-augmentation induces an isomorphism $\epsilon_\lambda : H_\lambda \mathfrak{F}_\lambda \sigma_\mu \xrightarrow{\cong} Z$. Tensoring with σ_μ and taking the direct sum over all $\sigma_\mu \in \mathfrak{F}_\mu$, we obtain the isomorphism

$$\epsilon_\lambda : H_\lambda D_{\lambda\mu} \xrightarrow{\cong} D_\mu.$$

We say that $\mathfrak{F}$ is (λ, μ, ν)-*independent* if for each $\sigma_\nu \in \mathfrak{F}_\nu$, the facet

$$\mathfrak{F}_{\lambda\mu} \sigma_\nu = \mathfrak{F}_\lambda \sigma_\nu \tilde{\otimes} \mathfrak{F}_\mu \sigma_\nu.$$

The definition is symmetrical in the first two coordinates λ and μ.

We are now in a position to prove the key lemmas. Roughly speaking they give permission to 'excise' the acyclic facets, firstly from the complementary degree of the spectral sequence, and secondly from the filtering degree.

Lemma 5. *If $\mathfrak{F}$ is $(\lambda, \mu\nu)$-acyclic then $\epsilon_\lambda : E_{\nu,\lambda\mu} \xrightarrow{\cong} E_{\nu,\mu}$, etc.*

Proof. The hypothesis implies that $\epsilon_\lambda : H_\lambda D_{\lambda\mu\nu} \xrightarrow{\cong} D_{\mu\nu}$. Therefore the spectral sequence

$$E_{\mu,\lambda} D_{\lambda\mu\nu} : H_\mu H_\lambda D_{\lambda\mu\nu} \Rightarrow H_{\lambda\mu} D_{\lambda\mu\nu}$$

collapses because the E^1 term is concentrated on the axis $n_\lambda = 0$. Hence

$$H_{\lambda\mu} D_{\lambda\mu\nu} \xrightarrow{\cong} H_\mu H_\lambda D_{\lambda\mu\nu} \xrightarrow[\epsilon_\lambda]{\cong} H_\mu D_{\mu\nu}.$$

Therefore
$$H_\nu H_{\lambda\mu} D_{\lambda\mu\nu} \xrightarrow[\epsilon_\lambda]{\cong} H_\nu H_\mu D_{\mu\nu}.$$

In other words ϵ_λ induces an isomorphism between the E^2 terms of the two spectral sequences concerned in the lemma, and consequently an isomorphism between the entire spectral sequences.

Lemma 6. *If $\epsilon_\nu : H_\nu H_\lambda D_{\lambda\mu\nu} \xrightarrow{\cong} H_\lambda D_{\lambda\mu}$ then $\epsilon_\nu : E_{\mu\nu,\lambda} \xrightarrow{\cong} E_{\mu,\lambda}$, etc.*

Proof. The spectral sequence

$$E_{\mu,\nu}(H_\lambda D_{\lambda\mu\nu}) : H_\mu H_\nu H_\lambda D_{\lambda\mu\nu} \Rightarrow H_{\mu\nu} H_\lambda D_{\lambda\mu\nu}$$

collapses, since by hypothesis the E^1 term is concentrated on the axis $n_\nu = 0$. Therefore

$$H_{\mu\nu} H_\lambda D_{\lambda\mu\nu} \xrightarrow{\cong} H_\mu H_\nu H_\lambda D_{\lambda\mu\nu} \xrightarrow[\epsilon_\nu]{\cong} H_\mu H_\lambda D_{\lambda\mu}.$$

The composition is an isomorphism between the E^2 terms of the two given spectral sequences, which is sufficient to prove the lemma.

Lemma 7. *If $\mathfrak{F}$ is (μ, ν)-acyclic and (λ, μ, ν)-independent, then*

$$\epsilon_\mu : E_{\mu\nu,\lambda} \xrightarrow{\cong} E_{\nu,\lambda}, \textit{ etc.}$$

Proof. By the independence, $\mathfrak{F}_{\lambda\mu}\sigma_\nu = \mathfrak{F}_\lambda\sigma_\nu \tilde{\otimes} \mathfrak{F}_\mu\sigma_\nu$ for all $\sigma_\nu \in \mathfrak{F}_\nu$. Therefore

$$H_\lambda \mathfrak{F}_{\lambda\mu}\sigma_\nu = H_\lambda \mathfrak{F}_\lambda\sigma_\nu \tilde{\otimes} \mathfrak{F}_\mu\sigma_\nu$$

because $\mathfrak{F}_\mu\sigma_\nu$ is free. The μ-augmentation induces $H_\mu \mathfrak{F}_\mu\sigma_\nu \xrightarrow{\cong} Z$ by acyclicity, and therefore

$$\epsilon_\mu : H_\mu H_\lambda \mathfrak{F}_{\lambda\mu}\sigma_\nu \xrightarrow{\cong} H_\lambda \mathfrak{F}_\lambda\sigma_\nu$$

by the universal coefficient theorem. Tensoring this equation by σ_ν and summing over $\sigma_\nu \in \mathfrak{F}_\nu$ we have

$$\epsilon_\mu : H_\mu H_\lambda D_{\lambda\mu\nu} \xrightarrow{\cong} H_\lambda D_{\lambda\nu}.$$

Now apply Lemma 6 with μ and ν interchanged.

In two of the applications (§§ 5 and 6) we shall need the conclusion of Lemma 7 without having the advantage of (λ, μ, ν)-independence, so it is necessary to develop a more delicate condition. For want of a better name we shall call a facing relation having this property (λ, μ, ν)-excisable, and Lemma 8 ensures that the property is sufficient for the purpose. The conditions (ν, μ)-acyclic and (λ, ν, μ)-independent do in fact imply (λ, μ, ν)-excisable, so that Lemma 8 implies Lemma 7. However, the proof of Lemma 7 was so simple compared with the formation and proof of Lemma 8 that it was worth giving separately. For the reader's peace of mind in what follows, it is admitted that in the applications each of λ, μ, ν is comprised of a single suffix, although of course this is immaterial to the proof.

The category $L^{\succ}$

Let L be a geometric chain complex. In particular L is a category. Let $L^{\succ}$ be the *derived* category of L. That is to say an *object* ω of $L^{\succ}$ is a map of L, or, in other words, $\omega = (\sigma, \tau)$ where $\sigma \succ \tau$ in L. And a *map* of $L^{\succ}$ is a relation $\omega \succ \omega'$, which is defined by $\sigma \succ \sigma', \tau \succ \tau'$, where $\omega = (\sigma, \tau)$, $\omega' = (\sigma', \tau')$.

We remark that $L^{\succ}$ is *not*† a facing relation on $K \otimes K$ because the facing condition is not satisfied. Therefore $L^{\succ}$ does not give rise to a differential group structure, but has only category structure. One might say that it was a notion more geometric than algebraic in flavour. However the category structure of $L^{\succ}$ is useful. We can define the subcategories $\bar{\omega} = \{\omega';\ \omega \succ \omega'\}$ and $\operatorname{st}\omega = \{\omega';\ \omega' \succ \omega\}$. If K is another geometric chain complex we can define a facing relation on the category $K \otimes L^{\succ}$ satisfying the facing condition. The facets in K will be subcomplexes, while those in $L^{\succ}$ will be only subcategories.

† Nor is $L^{\succ}$ a mixed facing relation as in (5) because the latter is contravariant in σ and covariant in τ, while $L^{\succ}$ is covariant in both σ and τ.

Acyclic carrier functors

The motivation behind the next definition is by no means obvious. Therefore let us first give the definition, and then a geometrical example. We do not wish to dwell on the example, nor verify the properties, for these are proved in a generalization of the example in Lemma 9; we only include it for the benefit of those readers who do not like to proceed without some kind of intuition.

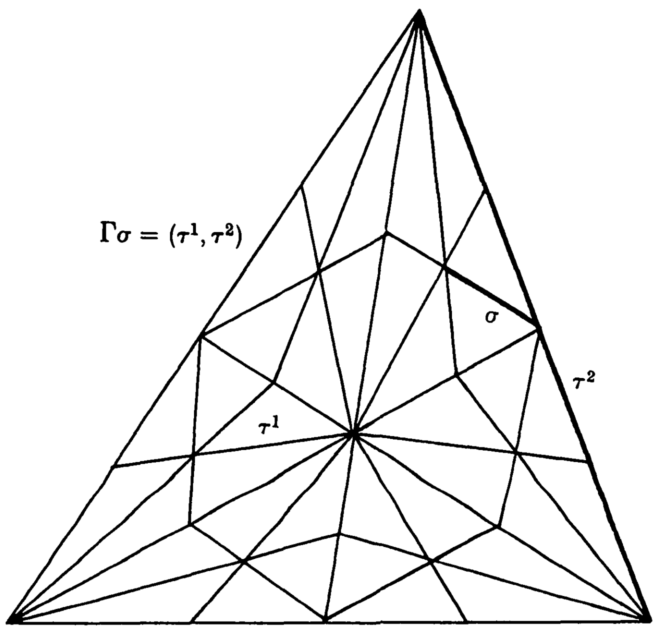

Fig. 1

Definition. If $\mathfrak{F}$ is a facing relation between K and L, we say that a functor $\Gamma : K \to L^{\succ}$ is an *acyclic functor carrying* $\mathfrak{F}$ if

(i) $\sigma \otimes \tau \in \mathfrak{F}$ implies $\Gamma\sigma \succ (\tau, \tau)$, and

(ii) for each triple $\tau^1 \succ \tau^2 \succ \tau$ in L, the subcomplex of K

$$\mathfrak{F}\tau \cap \Gamma^{-1}\overline{(\tau^1, \tau^2)} = \{\sigma;\ \sigma \otimes \tau \in \mathfrak{F} \quad \text{and} \quad (\tau^1, \tau^2) \succ \Gamma\sigma\}$$

is acyclic.

Example. Suppose that L triangulates a manifold. Each simplex $\tau \in L$ has a dual cell τ^*. Let K be the second derived complex of L. Let the facing relation on $K \otimes L$ be given by $|\sigma| \subset |\tau^*|$. Let $\Gamma : K \to L^{\succ}$ be given by $\Gamma\sigma = (\tau^1, \tau^2)$, where $|\sigma| \subset |\tau^1|$, and τ^2 is the smallest face of τ^1 meeting $|\overline{\mathrm{st}\,\sigma}|$. (See Fig. 1.)

Definition. Let $\mathfrak{F}$ be a multiple facing relation. We say that $\mathfrak{F}$ is (λ, μ, ν)-*excisable* if there exists an acyclic functor $\Gamma : K_\nu \to K_\mu^{\succ}$ carrying $\mathfrak{F}_{\mu\nu}$, and a facing relation $\mathfrak{F}^{\succ}$ between K_λ and $K_\mu^{\succ}$, such that for each $\sigma_\mu \in \mathfrak{F}_\mu$ and $\sigma_\mu \tilde{\otimes} \sigma_\nu \in \mathfrak{F}_{\mu\nu}$ there are chain equivalent inclusions between the facets:

$$\mathfrak{F}_\lambda \sigma_\mu \subset \mathfrak{F}_\lambda^{\succ}(\sigma_\mu, \sigma_\mu), \quad \mathfrak{F}_\lambda(\sigma_\mu \tilde{\otimes} \sigma_\nu) \subset \mathfrak{F}_\lambda^{\succ} \Gamma \sigma_\nu.$$

LEMMA 8. *If $\mathfrak{F}$ is (λ, μ, ν)-excisable then $\epsilon_\nu : E_{\mu\nu,\lambda} \xrightarrow{\cong} E_{\mu,\lambda}$, etc.*

Proof. It suffices to show that for each $\sigma_\mu \in \mathfrak{F}_\mu$, the ν-augmentation induces an isomorphism

$$\epsilon_\nu : H_\nu H_\lambda \mathfrak{F}_{\lambda\nu} \sigma_\mu \xrightarrow{\cong} H_\lambda \mathfrak{F}_\lambda \sigma_\mu;$$

for then tensoring by σ_μ and summing over $\sigma_\mu \in \mathfrak{F}_\mu$, and applying Lemma 6 gives the result.

Therefore let σ_μ be fixed in $\mathfrak{F}_\mu$. In the category $K_\mu^{\succ}$, let $\omega^0 = (\sigma_\mu, \sigma_\mu)$. Well-order $\omega^0, \omega^1, \omega^2, \ldots$ the objects of $\operatorname{st}\omega^0$ in such a way that if $\omega^i \succ \omega^j$ then $i \geqslant j$. This is possible since the partial ordering $\succ$ has the descending chain condition; or, explicitly, after ω^0 we can first order those pairs of cells of dimensions $(p+1, p)$ where p is the dimension of σ_μ, and then successively those pairs of dimensions $(p+1, p+1), (p+2, p), \ldots$, and so on.

We use this well-ordering to construct a (possibly transfinite) filtration Q^i of the facet $\mathfrak{F}_\nu \sigma_\mu$. First notice that $\Gamma \mathfrak{F}_\nu \sigma_\mu \subset \operatorname{st}\omega^0$ by the first property of an acyclic carrier functor. Let P^i $(i \geqslant 0)$ be the subcomplex of $\mathfrak{F}_\nu \sigma_\mu$ which is the inverse image under Γ of $\overline{\omega^i}$; alternatively we may write

$$P^i = \{\sigma_\nu;\ \sigma_\mu \tilde{\otimes} \sigma_\nu \in \mathfrak{F}_{\mu\nu} \quad \text{and} \quad \omega^i \succ \Gamma \sigma_\nu\}.$$

By the second property of an acyclic carrier functor P^i is acyclic. Let $Q^j = \bigcup_{i<j} P^i$ $(j \geqslant 1)$, and let $P^{i,j} = P^i \cap Q^j$. We next show that the $P^{i,j}$ and the Q^j are also acyclic. For suppose that some $P^{i,j}$ were not acyclic, and assume that i is the least ordinal for which this is so, and that j is least with respect to this i. Then $i \geqslant j$, otherwise $P^{i,j} = P^i$. Also $j > 1$, otherwise $j = 1$ and $P^{i,j} = Q^1 = P^0$, since all the P^i contain P^0. If j is an ordinal with predecessor, and if $P^{j-1} \subset P^i$, then a contradiction is achieved by using the Mayer–Vietoris Theorem, for

$$P^{i,j} = P^{i,j-1} \cup P^{j-1}, \quad P^{j-1,j-1} = P^{i,j-1} \cap P^{j-1},$$

and the three complexes $P^{j-1}, P^{i,j-1}, P^{j-1,j-1}$ are acyclic by hypothesis. On the other hand, if $P^{j-1} \not\subset P^i$, then $P^{i,j} = P^{i,j-1}$ by our choice of well-ordering. Finally if j is a limit ordinal a contradiction results from the fact that $P^{i,j}$ is the direct limit of the acyclic subcomplexes $P^{i,k}$, $k < j$, and the direct limit of acyclic complexes is acyclic because direct limits are exact, and so the homology of the limit is the limit of the homologies which is trivial.

To prove that the Q^j are acyclic, we proceed in much the same manner. Suppose Q^j to be the first not acyclic. Since $Q^1 = P^0$ the induction begins with $j > 1$. If j has a predecessor, $Q^j = Q^{j-1} \cup P^{j-1}$ and the three complexes $Q^{j-1}, P^{j-1}, P^{j-1,j-1}$ are acyclic. If j has no predecessor, Q^j is the direct limit of the acyclic Q^k, $k<j$. The contradiction in either case proves that all the Q^j are acyclic. In particular, if J is the first ordinal not in the well-ordering then $\mathfrak{F}_\nu \sigma_\mu = Q^J$ is acyclic.

We now pull the filtration back by means of the λ-augmentation to a filtration of the facet $\mathfrak{F}_{\lambda\nu}\sigma_\mu$: let R^j be the inverse image of Q^j under the homomorphism $\epsilon_\lambda : \mathfrak{F}_{\lambda\nu}\sigma_\mu \to \mathfrak{F}_\nu \sigma_\mu$. For any $\sigma_\nu \in Q^j$,

$$\mathfrak{F}_\lambda(\sigma_\mu \tilde{\otimes} \sigma_\nu) \subset \mathfrak{F}_\lambda \sigma_\mu.$$

Tensoring with σ_ν and summing over $\sigma_\nu \in Q^j$, the inclusions induce a homomorphism

$$R^j \to \mathfrak{F}_\lambda \sigma_\mu \tilde{\otimes} Q^j.$$

Taking homology with respect to λ,

$$H_\lambda R^j \to H_\lambda \mathfrak{F}_\lambda \sigma_\mu \tilde{\otimes} Q^j$$

since Q^j is free. Taking homology with respect to ν, we have, by the universal coefficient theorem since Q^j is an acyclic subcomplex of K_ν,

$$\theta^j : H_\nu H_\lambda R^j \to H_\lambda \mathfrak{F}_\lambda \sigma_\mu.$$

We have given this last homomorphism a name because we want to show inductively that θ^j is an isomorphism for all j.

First consider the case $j = 1$. If $\sigma_\nu \in Q^1 = P^0$, then

$$H_\lambda \mathfrak{F}_\lambda(\sigma_\mu \tilde{\otimes} \sigma_\nu) \xrightarrow{\cong} H_\lambda \mathfrak{F}_\lambda \sigma_\mu$$

is an isomorphism by the excisability. Tensoring by σ_ν and summing over $\sigma_\nu \in Q^1$,

$$H_\lambda R^1 \xrightarrow{\cong} H_\lambda \mathfrak{F}_\lambda \sigma_\mu \tilde{\otimes} Q^1,$$

and

$$\theta^1 : H_\nu H_\lambda R^1 \xrightarrow{\cong} H_\lambda \mathfrak{F}_\lambda \sigma_\mu.$$

Suppose that some θ^j is not an isomorphism, and assume that j is the least such. If j has a predecessor there is a commutative diagram induced by inclusions:

$$\begin{array}{ccc} H_\nu H_\lambda R^{j-1} & \xrightarrow[\theta^{j-1}]{\cong} & H_\lambda \mathfrak{F}_\lambda \sigma_\mu \\ \downarrow & & \downarrow 1 \\ H_\nu H_\lambda R^j & \xrightarrow[\theta^j]{\cong} & H_\lambda \mathfrak{F}_\lambda \sigma_\mu \end{array}.$$

Now if $\sigma_\nu \in Q^j - Q^{j-1}$ then $\Gamma\sigma_\nu = \omega^j$, and excisability gives an isomorphism

$$H_\lambda \mathfrak{F}_\lambda(\sigma_\mu \tilde{\otimes} \sigma_\nu) \xrightarrow{\cong} H_\lambda \mathfrak{F}_\lambda^* \omega^j,$$

so there is an exact sequence

$$0 \to H_\lambda R^{j-1} \to H_\lambda R^j \to H_\lambda \mathfrak{F}_\lambda^\times \omega^j \otimes Q^j/Q^{j-1} \to 0.$$

But $H_\nu(Q^j/Q^{j-1}) = 0$ because the Q^j are acyclic, and so by the relative exact sequence, inclusion induces an isomorphism

$$H_\nu H_\lambda R^{j-1} \xrightarrow{\cong} H_\nu H_\lambda R^j.$$

Hence the commutative diagram above gives θ^j an isomorphism, contradicting our supposition. If j has no predecessor, θ^j is the direct limit of a directed system of isomorphisms, and is therefore an isomorphism.

We have now shown that all the θ^j are isomorphisms, and in particular when $j = J$,

$$H_\nu H_\lambda \mathfrak{F}_{\lambda\nu}\sigma_\mu \xrightarrow{\cong} H_\lambda \mathfrak{F}_\lambda \sigma_\mu.$$

But this is none other than the isomorphism induced by ν-augmentation that we wished to prove, because the ν-augmentation can be factored into two homomorphisms,

$$\mathfrak{F}_{\lambda\nu}\sigma_\mu \to \mathfrak{F}_\lambda \sigma_\mu \otimes \mathfrak{F}_\nu \sigma_\mu \to \mathfrak{F}_\lambda \sigma_\mu,$$

of which the first induces the isomorphism above while the second induces the identity isomorphism on $H_\lambda \mathfrak{F}_\lambda \sigma_\mu$. Lemma 8 is proved.

4. Leray theory

We now link up the Čech dihomology theory with the sheaf theory of Leray (**3**). We recall the following

Definition. Let X be a topological space and R a ring. The *Alexander presheaf* $\mathfrak{L} = \mathfrak{L}(X; R)$ on X over R is defined by assigning

(i) to each open set U of X the normalized Alexander cochain complex $\Lambda^N(U; R)$ of U over R, and

(ii) if $U \supset U'$, the restriction homomorphism $\Lambda^N(U; R) \to \Lambda^N(U'; R)$.

If $f: X \to Y$ is a continuous map, then $f\mathfrak{L}$ is a multiplicative graded differential presheaf on Y. If β is a covering of Y, the cochain complex of Y with respect to β with coefficients in $f\mathfrak{L}$ is a double complex. Filtering with respect to the grading of the nerve of β, we obtain a spectral ring, and taking direct limits over β gives the Leray spectral ring.

THEOREM 2. *There is a canonical isomorphism between the Čech and Leray spectral rings of a continuous map.*

COROLLARY. *There is a natural equivalence between the Čech and Leray spectral functors on the category $\mathfrak{M}$ of continuous maps.*

Proof of the Theorem. Let R be the coefficient ring, and let $f: X \to Y$ be the continuous map. Let α, β be coverings of X, Y such that α refines

$f^{-1}\beta$. Consider the following triple facing relation on $K_\lambda \otimes K_\mu \otimes L_\nu$, where

$K_\lambda = N(\alpha)$, the nerve of α,

$K_\mu = V(X, \alpha)$, the Vietoris complex of α-small simplexes of X (see (8)),

$L_\nu = N(\beta)$, the nerve of β.

$$\mathfrak{F} = \{\sigma_\lambda \otimes \sigma_\mu \otimes \tau_\nu;\ \sigma_\mu \subset \sup \sigma_\lambda \cap f^{-1}(\sup \tau_\nu)\}.\ \text{(See Fig. 2.)}$$

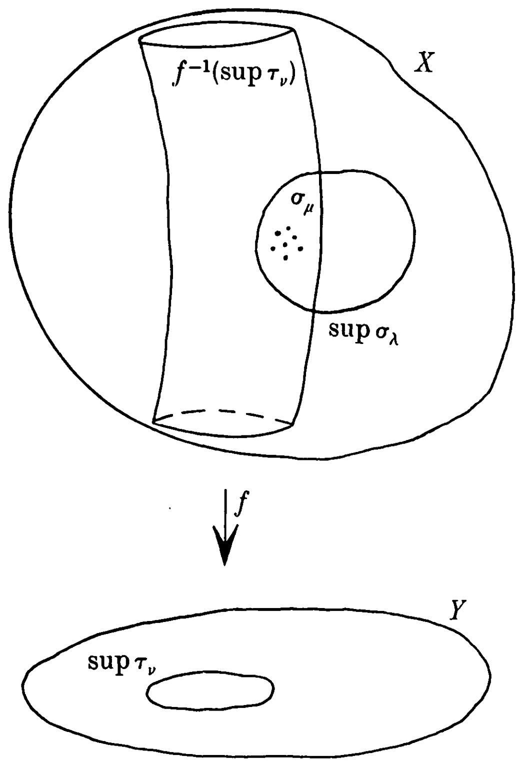

FIG. 2

Clearly $\mathfrak{F}$ satisfies the facing condition because Čech supports expand while Vietoris simplexes shrink when passing to faces. Also $\mathfrak{F}$ has the properties:

(1) $\mathfrak{F}$ is $(\mu, \lambda\nu)$-acyclic, because if $\sigma_\lambda \otimes \tau_\nu \in \mathfrak{F}_{\lambda\nu}$, then $\sup \sigma_\lambda$ meets $f^{-1}(\sup \tau_\nu)$ and the facet

$$\mathfrak{F}_\mu(\sigma_\lambda \otimes \tau_\nu) = V(\sup \sigma_\lambda \cap f^{-1}(\sup \tau_\nu))$$

is an acyclic Vietoris cone.

(2) $\mathfrak{F}$ is $(\lambda, \mu\nu)$-acyclic, because if $\sigma_\mu \otimes \tau_\nu \in \mathfrak{F}_{\mu\nu}$, then $\sigma_\mu \subset f^{-1}(\sup \tau_\nu)$ and the facet

$$\mathfrak{F}_\lambda(\sigma_\mu \otimes \tau_\nu) = \{\sigma_\lambda;\ \sup \sigma_\lambda \supset \sigma_\mu\}$$

is non-empty since σ_μ is α-small, and is an acyclic Čech cone.

(3) $\mathfrak{F}$ is functorial on $\mathfrak{M}_{\text{cov}}$. For let $(\phi, \psi, \phi_\alpha, \psi_\beta) : (f, \alpha, \beta) \to (f', \alpha', \beta')$ be a map of $\mathfrak{M}_{\text{cov}}$, and let $\sigma_\lambda \otimes \sigma_\mu \otimes \tau_\nu \in \mathfrak{F}(f, \alpha, \beta)$; then

$$\begin{aligned}\phi\sigma_\mu &\subset \phi(\sup \sigma_\lambda \cap f^{-1}(\sup \tau_\nu)) \\ &\subset \phi \sup \sigma_\lambda \cap (f')^{-1} \psi(\sup \tau_\nu) \\ &\subset \sup \phi_\alpha \sigma_\lambda \cap (f')^{-1} \sup \psi_\beta \tau_\nu.\end{aligned}$$

Therefore $\phi_\alpha \sigma_\lambda \otimes \phi\sigma_\mu \otimes \psi_\beta \tau_\nu \in \mathfrak{F}(f', \alpha', \beta')$.

Applying Lemma 5 to properties (1) and (2), we have isomorphisms

$$E_{\nu,\lambda} \xleftarrow[\epsilon_\mu]{\cong} E_{\nu,\lambda\mu} \xrightarrow[\epsilon_\lambda]{\cong} E_{\nu,\mu}.$$

Passing to the spectral rings over R, we have isomorphisms

$$E^{\nu,\lambda} \xrightarrow[\epsilon^\mu]{\cong} E^{\nu,\lambda\mu} \xleftarrow[\epsilon^\lambda]{\cong} E^{\nu,\mu}.$$

Since the isomorphisms are induced by augmentations, which are functorial, there is by (3) a natural equivalence between the contravariant functors

$$E^{\nu,\lambda} \cong E^{\nu,\mu} : \mathfrak{M}_{\text{cov}} \to \mathfrak{E}^*,$$

where $\mathfrak{E}^*$ is the category of spectral rings. Therefore there is a natural equivalence between the limit functors

$$\lim_{\alpha,\beta} E^{\nu,\lambda} \cong \lim_{\alpha,\beta} E^{\nu,\mu} : \mathfrak{M} \to \mathfrak{E}^*.$$

Now the facing relation $\mathfrak{F}_{\lambda\nu}$ is none other than the Čech facing relation $f(\sup \sigma_\lambda) \cap \sup \tau_\nu \neq \emptyset$, so the functor $\lim_{\alpha,\beta} E^{\nu,\lambda}$ is the Čech spectral ring functor of Theorem 1. It remains to identify $\lim_{\alpha,\beta} E^{\nu,\mu}$ with Leray's spectral ring.

The first thing to notice is that in taking direct limits over all pairs α, β we may first take limits over α and then over β (it is important to do it in this order because α has to refine $f^{-1}\beta$). The next thing to notice is that

$$\lim_\alpha (E^{\nu,\mu} D^{\mu\nu}) \cong E^{\nu,\mu} \left(\lim_\alpha D^{\mu\nu}\right)$$

since the direct limit functor is exact. Now $D_{\mu\nu} = C(L_\nu; \mathfrak{F}_\mu \cdot)$, the chain complex of L_ν with coefficients in the covariant stack of facets $\{\mathfrak{F}_\mu \tau_\nu\}$. Meanwhile†

$$D^{\mu\nu} = D_{\mu\nu} \pitchfork R = C^\cdot (L_\nu; \mathfrak{F}_\mu^\cdot)$$

the cochain complex of L_ν with coefficients in the contravariant stack $\{\mathfrak{F}_\mu \tau_\nu \pitchfork R\}$.

Fix τ_ν for the moment, and let $U = f^{-1}(\sup \tau_\nu)$. Then $\mathfrak{F}_\mu \tau_\nu = V(U, \alpha)$, the α-small Vietoris complex of U. We have the split exact sequence

$$0 \to \mathfrak{F}_\mu \tau_\nu \to V(U) \to V(U)/V(U, \alpha) \to 0,$$

† As in (8) we use $A \pitchfork B$ for Hom (A, B).

which remains split exact under the functor $\pitchfork R$,

$$0 \leftarrow \mathfrak{F}_\mu \tau_\nu \pitchfork R \leftarrow \Lambda(U; R) \leftarrow \Lambda(U, \alpha; R) \leftarrow 0,$$

where $\Lambda(U; R)$ is the Alexander cochain complex of U over R, and $\Lambda(U, \alpha; R)$ is the subcomplex of cochains which vanish on all α-small simplexes. Taking direct limits over α of this sequence, we observe that since $\lim_\alpha \Lambda(U, \alpha; R)$ is the subcomplex of cochains with zero support,

$$\lim_\alpha (\mathfrak{F}_\mu \tau_\nu \pitchfork R) = \Lambda^N(U; R),$$

the normalized Alexander cochain complex of U over R. But this is the complex assigned to U by the Alexander presheaf $\mathfrak{L}$, and to $\sup \tau_\nu$ by the presheaf $f\mathfrak{L}$. Moreover if $\tau_\nu{}^1 \succ \tau_\nu{}^2$ the homomorphism $\mathfrak{F}_\mu \tau_\nu{}^2 \pitchfork R \to \mathfrak{F}_\mu \tau_\nu{}^1 \pitchfork R$ of the stack $\mathfrak{F}_\mu^{\cdot}$ is induced by inclusion, and therefore in the limit is the same as the restriction homomorphism of the presheaf $f\mathfrak{L}$. Summarizing, we have shown that

$$\lim_\alpha D^{\mu\nu} = C^{\cdot}(L_\nu; f\mathfrak{L})$$

the cochain complex of Y with respect to β with coefficients in $f\mathfrak{L}$. Passing to the spectral ring and taking limits over β identifies $\lim_{\alpha,\beta} E^{\nu,\mu}$ with the Leray spectral ring. Both Theorem 2 and Corollary are proved.

5. Simplicial theory

Throughout this section we shall assume that we are given a fixed simplicial map between two finite oriented simplicial complexes K and L. Since it is necessary to distinguish between the simplicial map and the underlying continuous map $f: X \to Y$ between the polyhedra, let us denote the simplicial map by $f^{(0)}: K \to L$. It is natural to ask whether or not the simplicial spectral sequence $E(f^{(0)})$, as defined at the beginning of § 2, is isomorphic to the Čech sequence $\check{E}(f)$. The answer is in fact *not*, for the technical reason that the closures of stars of simplexes are too big as the example at the end of this section shows. It transpires that if we pass to the second derived map $f^{(2)}: K^{(2)} \to L^{(2)}$ between the second derived complexes then we do get the 'correct' answer, as not infrequently happens when working with simplicial complexes. In Theorem 3 we establish a canonical isomorphism between the simplicial $E(f^{(2)})$ and the Čech $\check{E}(f)$, *etc.* This on the one hand proves the topological invariance of $E(f^{(2)})$, and on the other furnishes a means of computing $\check{E}(f)$. If we are only interested in computing the E^2 term of the spectral sequence (and not the d^r's, $r \geqslant 2$) then we can do this satisfactorily from $f^{(0)}$ without having to pass to the derived complexes.

Derived maps

We define the *first derived map* $f^{(1)}$ of $f^{(0)}$ as follows. Let $L^{(1)}$ be the barycentric first derived complex of L. A vertex of $L^{(1)}$ is the barycentre $\hat{\tau}$ of some $\tau \in L$. Given $\sigma \in K$, with $f\sigma = \tau$ say, lift $\hat{\tau}$ to an interior point $\hat{\sigma}$ of σ, such that $\hat{\sigma} \in f^{-1}(\hat{\tau})$, and define $f^{(1)}\hat{\sigma} = \hat{\tau}$. If σ happens to be mapped non-degenerately, then $\hat{\sigma}$ coincides with the barycentre of σ, but otherwise it may not. The points $\hat{\sigma}$ form the vertices of a triangulation $K^{(1)}$ of X that, qua abstract complex, is the first derived of K. The function $f^{(1)}$ already defined on the vertices of $K^{(1)}$ determines a simplicial map of $f^{(1)}: K^{(1)} \to L^{(1)}$, and completes the definition. By induction we define the sequence of derived maps $f^{(s+1)} = (f^{(s)})^{(1)}$, $s = 0, 1, 2, \ldots$.

The point of this construction is that $f^{(1)}$ has the same underlying continuous map f as $f^{(0)}$. Therefore from the Čech point of view $f^{(1)}$ is a simplicial approximation to f with respect to the star coverings of $K^{(1)}$ and $L^{(1)}$. Consequently when computing $\check{E}(f)$ we can confine our attention to the sequence of derived simplicial maps, since the corresponding sequence of pairs of star coverings is cofinal in the directed set of all pairs of coverings of X, Y. That is why we required the complexes K and L to be finite.

In this section we shall have to take particular care to distinguish between the stars of different complexes, so when there is any likelihood of doubt we shall write $\operatorname{st}(\sigma, K)$ for the star of σ in K.

Fibres

If $y \in Y$ we shall call $f^{-1}y$ the *fibre* above y, even though f may not be a fibre map. If $B \subset Y$ and B is not a point, it is suggestive to refer to $f^{-1}B$ as the *solid fibre* above B. Of particular interest are the following four constructions. If $\tau \in L$, let $F\tau$ denote the solid fibre above $|\operatorname{st} \tau|$, and let $\hat{F}\tau$ denote the fibre above $\hat{\tau}$ the barycentre of τ. If $\tau^i \succ \tau^j$ let $F(\tau^i, \tau^j)$ be the closure of $F\tau^i$ in $F\tau^j$, and let $\hat{F}(\tau^i, \tau^j) = F(\tau^i, \tau^j) \cap \hat{F}\tau^j$. In particular $F(\tau, \tau) = F\tau$ and $\hat{F}(\tau, \tau) = \hat{F}\tau$. Whereas $\hat{F}(\tau^i, \tau^j)$ is the polyhedron underlying a subcomplex of $K^{(1)}$, $F(\tau^i, \tau^j)$ is not in general a polyhedron; in particular $F\tau$ is an open set of X. The linear contraction of $|\operatorname{st} \tau|$ to $\hat{\tau}$ can be lifted linearly to a deformation retraction of $F\tau$ onto $\hat{F}\tau$. If $\tau^i \succ \tau^j$ there is a similar deformation retraction of $F(\tau^i, \tau^j)$ onto $\hat{F}(\tau^i, \tau^j)$.

The stack $\mathfrak{H} = \mathfrak{H}_*(f)$, *etc.*

We associate with the simplicial map $f: K \to L$ a covariant graded stack $\mathfrak{H}_*(f) = \sum_p \mathfrak{H}_p(f)$ on L, defined below. Similarly we can define for arbitrary coefficients G, R a covariant stack $\mathfrak{H}_*(f;\, G)$ and a contravariant stack $\mathfrak{H}^*(f;\, R)$. Since we shall make much use of $\mathfrak{H}_*(f)$ in this section while f

stays the same, we shall abbreviate $\mathfrak{H}_*(f)$ to $\mathfrak{H}$. The definition runs:

(i) To each $\tau \in L$ assign the homology group $\mathfrak{H}\tau = H_*(F\tau)$.

(ii) If $\tau^i \succ \tau^j$ assign the homomorphism $\mathfrak{H}\tau^i \to \mathfrak{H}\tau^j$ induced by inclusion.

It does not matter whether we use Čech or singular homology in the definition, since both are equal because both are preserved under the deformation retraction $F\tau \to \hat{F}\tau$, and $\hat{F}\tau$, being a polyhedron, has unique computable homology. In practice, of course, we compute $\mathfrak{H}$ from the homology groups of the fibres $\hat{F}\tau$.

THEOREM 3. *If f is the underlying continuous map of the simplicial map $f^{(0)} : K \to L$, there is a canonical isomorphism between the simplicial spectral sequence $E(f^{(2)})$ and the Čech sequence $\check{E}(f)$, etc. The sequence runs*

$$H_q(L; \mathfrak{H}_p(f)) \underset{q}{\Longrightarrow} H_n(K).$$

COROLLARY 3.1. *The Čech sequence of a simplicial map is spectral.*

COROLLARY 3.2. *The Čech sequence of a simplicial map converges $E^r = E^\infty$ at $r = \max(p+1, q)$, where p is the maximum dimension of a fibre and q is the dimension of the image space.*

Proof of Theorem 3. By Lemma 4 we can compute the Čech sequence $\check{E}(f)$ using oriented nerves (rather than nerves). Therefore

$$\check{E}(f) = \lim \check{E}^0(f, \alpha, \beta),$$

the inverse limit taken over all pairs of coverings α, β of $|K|$, $|L|$, respectively, such that α refines $f^{-1}\beta$, where $\check{E}^0(f, \alpha, \beta)$ denotes the spectral sequence obtained using oriented nerves.

Let $\alpha^{(s)}, \beta^{(s)}$ denote the star coverings of $K^{(s)}, L^{(s)}$, respectively. We may choose oriented nerves of $\alpha^{(s)}, \beta^{(s)}$ to be identical with $K^{(s)}, L^{(s)}$; then $\check{E}^0(f, \alpha^{(s)}, \beta^{(s)})$ is identical with the simplicial sequence $E(f^{(s)})$, by the definition in § 2. As we have already observed, the set of pairs $\alpha^{(s)}, \beta^{(s)}$ is cofinal in the directed set of all pairs of coverings of $|K|, |L|$. Therefore to compute $\check{E}(f)$ it is sufficient to confine our attention to these pairs:

$$\check{E}(f) \xrightarrow{\cong} \lim_s E(f^{(s)}).$$

If $s \geqslant t$, let $\pi^{s,t} : E(f^{(s)}) \to E(f^{(t)})$ denote the appropriate homomorphism in the inverse system. We shall prove that for each $s \geqslant 2$, there is an isomorphism

$$\pi^s : E^2(f^{(s)}) \xrightarrow{\cong} H_*(L; \mathfrak{H})$$

such that for $s \geqslant t \geqslant 2$, the diagram

$$\begin{array}{ccc} E^2(f^{(s)}) & \xrightarrow{\pi^{s,t}} & E^2(f^{(t)}) \\ & {\scriptstyle \pi^s}\searrow \quad \swarrow{\scriptstyle \pi^t} & \\ & H_*(L; \mathfrak{H}) & \end{array}$$

is commutative. Therefore $\pi^{s,t}$ is an isomorphism on the E^2 terms and consequently on the whole spectral sequence. The inverse system is a sequence of isomorphisms for $s \geqslant 2$, so that the homomorphism of the limit term

$$\check{E}(f) \xrightarrow{\cong} E(f^{(2)})$$

is an isomorphism as desired. Moreover we have shown that the sequence runs as stated in the theorem. There remains the main burden of the proof, the definition of the isomorphisms π^s.

Let s be fixed for the moment, $\geqslant 2$. We shall use a triple facing relation on $K_\lambda \otimes L_\mu \otimes L_\nu$, where $K_\lambda = K^{(s)}$, $L_\mu = L$, and $L_\nu = L^{(s)}$. For a typical cell we shall write $\sigma_\lambda \otimes \tau_\mu \otimes \tau_\nu$ to remind ourselves that the σ's are in X and the τ's in Y. To express the facing relation we shall need a generalization to complexes of the notion of the dual cell of a simplex in a combinatorial manifold.

Definition. Let $\tau = b^0 b^1 \dots b^q$ be a simplex of L. Define the *dual* of τ by

$$\operatorname{dual} \tau = \overline{\operatorname{st}(b^0, K^{(1)})} \cap \dots \cap \overline{\operatorname{st}(b^q, K^{(1)})}.$$

The dual of τ is a cone with vertex at the barycentre of τ, and is therefore an acyclic subcomplex of $L^{(1)}$.

The facing relation is given by

$$\mathfrak{F} = \{\sigma_\lambda \otimes \tau_\mu \otimes \tau_\nu;\ f|\operatorname{st}\sigma_\lambda| \cap |\operatorname{st}\tau_\nu| \neq \emptyset \quad \text{and} \quad |\operatorname{dual}\tau_\mu| \supset |\tau_\nu|\}.$$

Clearly $\mathfrak{F}$ satisfies the facing condition because duals are closed and expand when passing to faces, and similarly stars expand. Consider the properties of $\mathfrak{F}$:

(1) $\mathfrak{F}$ is (μ, ν)-acyclic, because the facet $\mathfrak{F}_\mu \tau_\nu$ is merely the closure of the largest simplex whose dual contains $|\tau_\nu|$.

(2) $\mathfrak{F}$ is (λ, μ, ν)-independent, because, for a given τ_ν, the two independent conditions in the facing relation ensure that $\mathfrak{F}_{\lambda\mu}\tau_\nu = \mathfrak{F}_\lambda \tau_\nu \otimes \mathfrak{F}_\mu \tau_\nu$.

(3) $\mathfrak{F}$ is (λ, μ, ν)-excisable by the next lemma, Lemma 9. Therefore applying Lemmas 7 and 8 we have isomorphisms

$$E_{\nu,\lambda} \underset{\epsilon_\mu}{\overset{\cong}{\longleftarrow}} E_{\mu\nu,\lambda} \underset{\epsilon_\nu}{\overset{\cong}{\longrightarrow}} E_{\mu,\lambda}.$$

(4) The facing relation $\mathfrak{F}_{\lambda\nu}$ is none other than the simplicial facing relation of $f^{(s)}$, because every τ_ν is contained in some $|\operatorname{dual}\tau_\mu|$, and so μ-augmentation leaves us with only the first condition $f|\operatorname{st}\sigma_\lambda| \cap |\operatorname{st}\tau_\nu| \neq \emptyset$. Therefore $E_{\nu,\lambda} = E(f^{(s)})$.

(5) There is an isomorphism induced by inclusion

$$E^2_{\mu,\lambda} \xrightarrow{\cong} H_*(L; \mathfrak{H}).$$

For consider the facing relation $\mathfrak{F}_{\lambda\mu}$. Fix τ_μ for the moment. The facet $\mathfrak{F}_\lambda \tau_\mu$ is a subcomplex of $K_\lambda = K^{(s)}$, and triangulates the solid fibre over the subcomplex $T_\nu = f^{(s)} \mathfrak{F}_\nu \tau_\mu$ of $L_\nu = L^{(s)}$. In detail

$$T_\nu = \{\tau_\nu ; |\operatorname{dual} \tau_\mu| \cap |\overline{\operatorname{st} \tau_\nu}| \neq \emptyset\},$$

or in other words T_ν is the closed simplicial neighbourhood of $|\operatorname{dual} \tau_\mu|$ in L_ν. This is the point where it was essential to have chosen $s \geq 2$, for then $|T_\nu| \subset |\operatorname{st} \tau_\mu|$. Consequently $|T_\nu|$ is deformable onto $|\operatorname{dual} \tau_\mu|$, which, being a cone, is contractible to its vertex τ_μ. We may lift this to a deformation retraction of $|\mathfrak{F}_\lambda \tau_\mu|$ onto $\hat{F}\tau_\mu$. Therefore inclusions induce isomorphisms

$$H_*(\hat{F}\tau_\mu) \xrightarrow{\cong} H_\lambda \mathfrak{F}_\lambda \tau_\mu \xrightarrow{\cong} H_*(F\tau_\mu) = \mathfrak{H}\tau_\mu .$$

If $\tau_\mu{}^i \succ \tau_\mu{}^j$, inclusions induce a commutative diagram

$$\begin{array}{ccc} H_\lambda \mathfrak{F}_\lambda \tau_\mu{}^i & \xrightarrow{\cong} & \mathfrak{H}\tau_\mu{}^i \\ \downarrow & & \downarrow \\ H_\lambda \mathfrak{F}_\lambda \tau_\mu{}^j & \xrightarrow{\cong} & \mathfrak{H}\tau_\mu{}^j \end{array} .$$

The resulting isomorphism between the stacks $H_\lambda \mathfrak{F}_\lambda \xrightarrow{\cong} \mathfrak{H}$ on L_μ induces the isomorphism $E^2_{\mu,\lambda} \xrightarrow{\cong} H_*(L; \mathfrak{H})$ that we want.

Combining all the above properties of $\mathfrak{F}$ we are able to define π^s as the composition

$$\underbrace{E^2(f^{(s)}) = E^2_{\nu,\lambda} \xleftarrow{\cong} E^2_{\mu\nu,\lambda} \xrightarrow{\cong} E^2_{\mu,\lambda} \xrightarrow{\cong} H_*(L; \mathfrak{H})}_{\pi^s}.$$

If $s \geq t \geq 2$, we have the commutative diagram

$$\begin{array}{ccccccc} E^2_{\nu,\lambda}(f^{(s)}) & \xleftarrow{\cong} & E^2_{\mu\nu,\lambda}(f^{(s)}) & \xrightarrow{\cong} & E^2_{\mu,\lambda}(f^{(s)}) & \xrightarrow{\cong} & H_*(L; \mathfrak{H}) \\ \downarrow \pi^{s,t} & & \downarrow & & \downarrow & & \downarrow 1 \\ E^2_{\nu,\lambda}(f^{(t)}) & \xleftarrow{\cong} & E^2_{\mu\nu,\lambda}(f^{(t)}) & \xrightarrow{\cong} & E^2_{\mu,\lambda}(f^{(t)}) & \xrightarrow{\cong} & H_*(L; \mathfrak{H}) \end{array} ,$$

the right-hand square being commutative because it is induced by inclusions, and the two left-hand squares being commutative by the functorial quality of $\mathfrak{F}$ on the derived maps. This diagram gives the formula $\pi^s = \pi^t \pi^{s,t}$ that we wanted to prove. Therefore the definition of π^s and the proof of Theorem 3 are completed by the following lemma.

LEMMA 9. *$\mathfrak{F}$ is (λ, μ, ν)-excisable.*

Proof. We have to define an acyclic functor $\Gamma : L_\nu \to L_\mu^\succ$ carrying $\mathfrak{F}_{\mu\nu}$, and a facing relation $\mathfrak{F}^\succ$ between K_λ and $L_\mu^\succ$. The functor is a generalization (to non-manifolds) of the example in § 3.

Define $\Gamma\tau_\nu = (\tau_\mu{}^1, \tau_\mu{}^2)$, where $\tau_\mu{}^1$ is the unique simplex of L_μ whose interior contains $|\tau_\nu|$, and τ^2 is the unique smallest face meeting $|\overline{\text{st}\,\tau_\nu}|$. The latter exists because the condition $s \geqslant 2$ ensures that the closed stars of L_ν refine the open stars of L_μ, and $\tau_\mu{}^2$ is merely the join of those vertices of L_μ whose stars contain $|\overline{\text{st}\,\tau_\nu}|$. We now verify the two conditions required of an acyclic carrier functor.

(i) The closed simplicial neighbourhood T_ν of $|\text{dual}\,\tau_\mu|$ in L_ν is contained in $|\text{st}\,\tau_\mu|$. Therefore if $\tau_\mu \otimes \tau_\nu \in \mathfrak{F}_{\mu\nu}$, then $|\tau_\nu| \subset |\text{dual}\,\tau_\mu|$ and $|\text{st}\,\tau_\nu| \subset |\text{st}\,\tau_\mu|$. Hence all the vertices of τ_μ are vertices of $\tau_\mu{}^2$ by the above description of $\tau_\mu{}^2$. Therefore $\Gamma\tau_\nu = (\tau_\mu{}^1, \tau_\mu{}^2) \succ (\tau_\mu, \tau_\mu)$, as desired.

(ii) Given $\tau_\mu{}^1 \succ \tau_\mu{}^2 \succ \tau$, we have to show that

$$P_\nu = \{\tau_\nu;\ \tau_\mu \otimes \tau_\nu \in \mathfrak{F}_{\mu\lambda} \text{ and } (\tau_\mu{}^1, \tau_\mu{}^2) \succ \Gamma\tau_\nu\}$$

is acyclic. Let $0_\nu{}^1, 0_\nu{}^2$ be the subcomplexes of L_ν triangulating the intersections of $|\text{dual}\,\tau_\mu|$ with $|\tau_\mu{}^1|$, $|\tau_\mu{}^2|$ respectively. Both $|0_\nu{}^1|$ and $|0_\nu{}^2|$ are convex. Therefore P_ν, which is the closed simplicial neighbourhood of $0_\nu{}^2$ in $0_\nu{}^1$, is acyclic. An example of P_ν is shown shaded in Fig. 3.

Having established the functor Γ, we now turn our attention to the facing relation $\mathfrak{F}^{\succ}$. Recall that if $\omega_\mu = (\tau_\mu{}^1, \tau_\mu{}^2) \in L_\mu^{\succ}$, then $F\omega_\mu$ is the closure of $F\tau_\mu{}^1$ in $F\tau_\mu{}^2$. Define

$$\mathfrak{F}^{\succ} = \{\sigma_\lambda \otimes \omega_\mu;\ |\overline{\sigma_\lambda}| \subset F\omega_\mu,\ \sigma_\lambda \in K_\lambda,\ \omega_\mu \in L_\mu^{\succ}\}.$$

Therefore the facet $\mathfrak{F}_\lambda^{\succ}\omega_\mu$ is the largest subcomplex of K_λ contained in $F\omega_\mu$, and $|\mathfrak{F}_\lambda^{\succ}\omega_\mu|$ is in fact a deformation retract of $F\omega_\mu$. In the particular case when $\omega_\mu = (\tau_\mu, \tau_\mu)$ we already know that $|\mathfrak{F}_\lambda \tau_\mu|$ is contained in, and is a deformation retract of, $F(\tau_\mu, \tau_\mu) = F\tau_\mu$. Therefore there is a chain equivalent inclusion

$$\mathfrak{F}_\lambda \tau_\mu \subset \mathfrak{F}_\lambda^{\succ}(\tau_\mu, \tau_\mu).$$

Finally, to establish the other chain equivalent inclusion, suppose $\tau_\mu \otimes \tau_\nu \in \mathfrak{F}_{\mu\nu}$, and suppose $\Gamma\tau_\nu = \omega_\mu = (\tau_\mu{}^1, \tau_\mu{}^2)$. The facet $\mathfrak{F}_\lambda(\tau_\mu \otimes \tau_\nu)$ is the closure of the solid fibre above $|\text{st}\,\tau_\nu|$, and is therefore contained as a point set in $F\omega_\mu$, and as a subcomplex in $\mathfrak{F}_\lambda^{\succ}\omega_\mu$. To show that the inclusion

$$\mathfrak{F}_\lambda(\tau_\mu \otimes \tau_\nu) \subset \mathfrak{F}_\lambda^{\succ}\omega_\mu$$

is a chain equivalence, we present a polyhedron, homeomorphic to $\hat{F}\omega_\mu$, which is a deformation retract of both $|\mathfrak{F}_\lambda(\tau_\mu \otimes \tau_\nu)|$ and $F\omega_\mu$, and therefore also of $|\mathfrak{F}_\lambda^{\succ}\omega_\mu|$ because this is a deformation retract of $F\omega_\mu$. If $y \in |\overline{\text{st}\,\tau_\nu}| \cap |\tau_\mu{}^2|$, the linear contraction of $|\text{st}\,\tau_\mu{}^2|$ to y can be lifted linearly to a deformation retraction of $F\omega_\mu$ onto the polyhedron $F\omega_\mu \cap f^{-1}y$. Similarly the contraction of $|\overline{\text{st}\,\tau_\nu}|$ to y can be lifted to a deformation retraction of $|\mathfrak{F}_\lambda(\tau_\mu \otimes \tau_\nu)|$ onto the same polyhedron. The proof of Lemma 9 and Theorem 3 is complete.

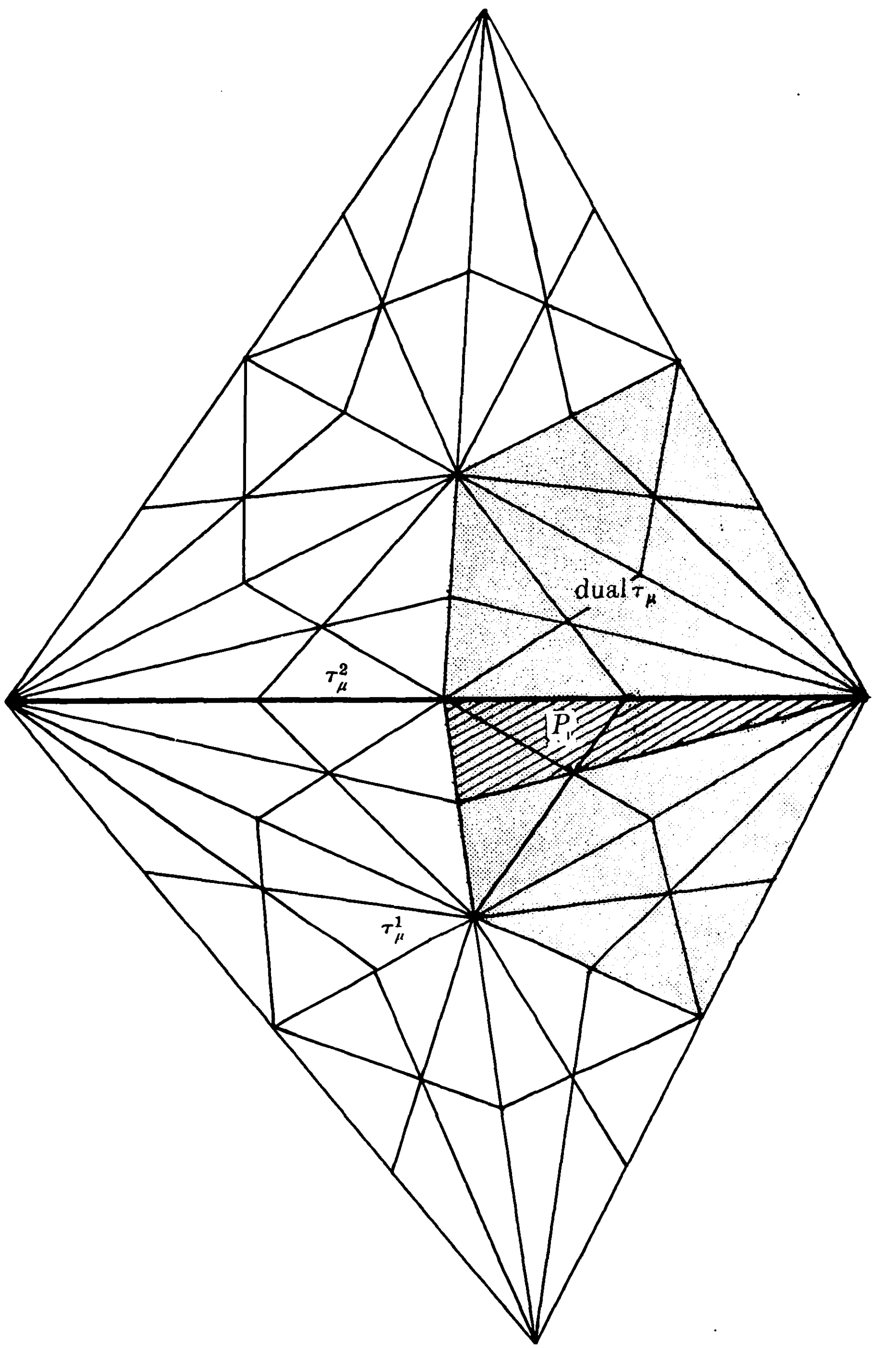

Fig. 3

Example. We give an example to show that it is necessary to pass to the second derived map to be sure of the correct computation of the Čech sequence.

Let $K = \dot{\sigma}$, the boundary of a 3-simplex $\sigma = a^0 a^1 a^2 a^3$. Let $L = \bar{\tau}$, a 1-simplex $\tau = b^0 b^1$. Let $f^{(0)} : K \to L$ be the simplicial map given by $fa^0 = fa^1 = b^0$, $fa^2 = fa^3 = b^1$. To compute $\mathfrak{H}$ we notice that the fibres above b^0, b^1 are contractible while that above $\hat{\tau}$ is a 1-sphere. Therefore the Čech spectral sequence $\check{E}(f)$ of the underlying continuous map converges at $\check{E}^2 = \check{E}^\infty$, with

$$\check{E}^2_{q,p}(f) = \begin{cases} Z, & p = q = 0 \quad \text{or} \quad p = q = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Meanwhile in the simplicial facing relation of $f^{(0)}$, the facets of b^0, b^1, τ all coincide with K, so that the simplicial spectral sequence $E(f^{(0)})$ also converges at $E^2 = E^\infty$, but with

$$E^2_{q,p}(f^{(0)}) = \begin{cases} Z, & p = 0, 2, \quad q = 0, \\ 0, & \text{otherwise.} \end{cases}$$

A small calculation shows that $E(f^{(1)})$ is also in error, so that $E(f^{(2)})$ is the first correct computation for $\check{E}(f)$.

Homotopic maps

The above example also illustrates that the Čech spectral sequence, although a topological invariant of the map, is not an invariant of the homotopy class of the map, as Leray pointed out for the spectral ring, in (**3**). For consider the constant map $g : K \to L$, that maps K to b^0, which is homotopic to f. A glance at the fibres will convince the reader that $\check{E}(g) \neq \check{E}(f)$; in fact $\check{E}(g)$ happens to be isomorphic to $E(f^{(0)})$ above. The difference may be analysed by noticing what has happened to the q-filtration of $H_*(K)$, and in particular to the filtration of a generator of $H_2(K)$. In passing from f to g the filtration has slipped from 1 to 0. The phenomenon may be interpreted geometrically by the fact that the 2-sphere, which represents a generating cycle of $H_2(K)$, is mapped by f to a 1-dimensional subset of $|L|$ and by g to a 0-dimensional subset. The dimension of the image of the sphere dictates the way in which it is fibred, and is thereby captured in the spectral sequence.

This observation suggests that we seek a more delicate equivalence relation between maps than homotopy. For example call two maps $f, g : X \to Y$ *isotopic* if $f = hg$, where h is a homeomorphism of Y onto itself that is isotopic to the identity. Then the spectral sequence $E(f)$ is an invariant of the isotopy class of f.

6. Computation theorem

As pointed out in the introduction, Theorem 3 is not much use for computation in fibre bundle theory, because the projection of the bundle is seldom simplicial. Therefore we introduce the notion of a *polyfibre map* between compact spaces, for which it is possible to compute the spectral sequences finitely. The definition is tailored on the one hand to make computation possible, and on the other to include the following examples of maps:

(i) Simplicial maps.

(ii) Fibre bundles with polyhedral fibre and base.

(iii) Ramified coverings of polyhedra.

(iv) The projection $M \to M/G$ of a manifold onto its orbit space M/G under the action of a Lie group G (provided G acts sufficiently smoothly).

(v) The projection from product to symmetric product of a polyhedron.

Since simplicial decompositions are uneconomical for computation, we shall use a cellular decomposition of the base of the polyfibre map, with the restriction that the cells must have non-singular boundaries, so that their stars can be contractible, and so that we can form derived complexes.

Cell complexes

Recall the definition. A *finite cell complex* L on a space Y is a finite collection $\{e_q^i\}$ of disjoint subsets called cells, such that

(i) $Y = \bigcup e_q^i$,

(ii) for each q-cell, $e = e_q^i$, there is a homeomorphism $B^q \to \bar{e}$ from the q-ball B^q onto the closure $\bar{e}$ of the cell, that maps the interior of the ball onto e,

(iii) the boundary, $\dot{e} = \bar{e} - e$, of a cell is an exact union of cells of lower dimension, which, together with e itself, are defined to be the faces of e.

Choose an orientation for each cell, and then the cell complex gives rise to a geometric chain complex, also denoted by L.

The barycentre $\hat{e}$ of a cell e is defined to be the image of the centre of the ball under the homeomorphism $B^q \to \bar{e}$. By joining the points $\{\hat{e}_q^i\}$ suitably, we can form the first derived complex $L^{(1)}$ of L, which is a simplicial complex triangulating Y, with the points $\{\hat{e}_q^i\}$ as vertices. Therefore Y is a polyhedron. Inductively we can define the tth derived $L^{(t)}$ of L.

Both the closure, $\bar{e} = \bigcup_{e > e^i} e^i$, of a cell, which is a closed subset of Y, and the star, $|\operatorname{st} e| = \bigcup_{e^i > e} e^i$, of the cell, which is an open subset of Y, are contractible to the barycentre $\hat{e}$. It is possible to define *duals* in a cell complex as in a simplicial complex:

$$\operatorname{dual} e = \bigcap_{e > e^i} \overline{\operatorname{st}(\hat{e}^i, L^{(1)})},$$

which is an acyclic subcomplex of $L^{(1)}$. The underlying point set, $|\operatorname{dual} e|$, is closed in Y, and is a cone with vertex $\hat{e}$, and is a deformation retract of $|\operatorname{st} e|$.

Given a continuous map $f: X \to Y$, we introduce the same fibre notation as in the last section: let $Fe = f^{-1}|\operatorname{st} e|$, and $\hat{F}e = f^{-1}\hat{e}$.

Definition of a polyfibre map

We say that a map $f: X \to Y$ is a *polyfibre map* if there exists a cell complex L on Y, such that for each cell $e \in L$:

(i) $\hat{F}e$ is a polyhedron.

(ii) There exists a retraction $\rho(e): Fe \to \hat{F}e$, which maps the fibre above any point in e *homeomorphically* onto $\hat{F}e$, and maps the fibre above any point in $|\operatorname{st} e|$ *onto* $\hat{F}e$, and which induces an isomorphism of Čech homology $\check{H}_*(Fe) \xrightarrow{\cong} \check{H}_*(\hat{F}e)$.

(iii) Given a point $x \in f^{-1} e$, define an e-rectangular neighbourhood of x to be an open set of X of the form $\rho(e)^{-1} U \cap f^{-1} V$, where U is open in $\hat{F}e$ and V is open in $|\operatorname{st} e|$: we require that the e-rectangular neighbourhoods of x form a base of neighbourhoods for x, for all $x \in X$.

We say that such a complex L is *associated* with the polyfibre map f. If L is associated with f, then the derived complexes of L are also.

Example. Let f be the orthogonal projection of a circle X onto a diameter Y. Then f is a polyfibre map with two types of fibre, single points and pairs of points. Triangulating Y as a 1-simplex provides an associated complex, for which the axioms are easy to verify.

All the examples mentioned at the beginning of this section can also be shown to be polyfibre maps.

Remarks. Notice that in a polyfibre map $f: X \to Y$ all the fibres are polyhedra, and the solid fibre above each cell is a product. However X itself need not be a polyhedron, since there is little restriction on the way in which the solid fibre above one cell is glued onto that above another cell. From (iii) one can show that X is compact. Y is compact because it is a polyhedron. Another important consequence of (ii) and (iii) is that f is an open map.

The stack $\mathfrak{H}$, etc.

Given a polyfibre map f and an associated complex L, we can define, as in the last section, the covariant stack $\mathfrak{H} = \mathfrak{H}_*(f) = \Sigma \mathfrak{H}_p(f)$ on L; to each cell $e \in L$ assign the Čech homology group $\mathfrak{H}e = \check{H}_*(Fe), \cong \check{H}_*(\hat{F}e)$ by hypothesis (ii) above, and if $e^i \succ e^j$ assign the homomorphism $\mathfrak{H}e^i \to \mathfrak{H}e^j$ induced by the inclusion $Fe^i \subset Fe^j$. Similarly define the stacks $\mathfrak{H}$, *etc.*

The computation facing relation

Suppose $f: X \to Y$ is a polyfibre map with an associated cell complex L on Y. Let α be a finite covering of X, and let $K = N^0(\alpha)$ be an oriented nerve of α. Define the *computation facing relation* $\tilde{\tilde{\mathfrak{F}}}$ between K and L by

$$\tilde{\tilde{\mathfrak{F}}} = \{\sigma \otimes e; f(\sup \sigma) \cap |\operatorname{dual} e| \neq \emptyset\}.$$

The facing condition is satisfied because both supports and duals expand on passing to faces. Filtering the resulting double complex with respect to q, the grading of L, we obtain a spectral sequence which we denote by $\tilde{\tilde{E}}(f, \alpha, L)$, *etc.* The advantage of this over the ordinary Čech facing relation is that fewer cells are involved, so that from the point of view of computation it is much more economical.

THEOREM 4. (*Computation theorem.*) *If $f: X \to Y$ is a polyfibre map, and L an associated cell complex on Y, then we can find a finite covering α of X, such that the Čech sequence $\check{E}(f)$ is isomorphic to $\tilde{\tilde{E}}(f, \alpha, L)$, etc. The sequence runs*

$$H_q(L; \mathfrak{H}_p(f)) \underset{q}{\Longrightarrow} H_n(X).$$

COROLLARY. *The Čech sequence of a polyfibre map is spectral.*

The rest of the section is devoted to the proof of Theorem 4. The pattern of proof is much the same as that of Theorem 3, except that in this case we have the additional complication of constructing α, and the need for rather careful handling of Čech theory. The covering α will consist of rectangular sets.

Construction of α

We introduce the notation: if γ is a covering of the solid fibre $f^{-1}V$ above a subset V of Y, and if $W \subset V$, let γ/W denote the covering cut down to the solid fibre above W.

Let $e^1, e^2, \ldots, e^m$ enumerate the cells of L in some order of increasing dimension. Let M^i be a simplicial complex triangulating the polyhedron $\hat{F}e^i$. Let $\gamma(e^i, s)$ be the inverse image under $\rho(e^i)$ of the star covering of the sth derived complex of M^i. Therefore $\gamma(e^i, s)$ is a finite covering of Fe^i by e^i-rectangular sets.

We shall define inductively two sequences of integers s^i, t^i $(i = 1, 2, \ldots, m)$, with t^i non-decreasing, and starting with $s^1 = 0$, $t^1 = 2$. Suppose that i is fixed for the moment, and s^j, t^j have been defined for $j < i$. We proceed to define s^i and t^i. If $j<i$, let A^j be the point set underlying the open simplicial neighbourhood of e^j in $L^{(t^j)}$, and let $B^j = \bigcup_{k \leqslant j} A^k$. Then B^j is an open subset of Y. Let $J^i = e^i - B^{i-1}$, which is a closed compact subset of e^i underlying a subcomplex of $L^{(t^{i-1})}$.

LEMMA 10. *Given $y \in J^i$, there is an integer s and a neighbourhood W of y contained in $|\operatorname{st} e^i|$, such that $\gamma(e^i, s)/W$ refines $\gamma(e^j, s^j)/W$ for all j, $e^i \succ e^j$.*

Proof. Since e^i is fixed for the lemma, we may abbreviate $\rho(e^i)$ to ρ. Let $x \in f^{-1}y$. For each $j, e^i \succ e^j$, choose a set Δ^j of $\gamma(e^j, s^j)$ containing x. By property (iii) of the polyfibre map, we can choose an e^i-rectangular set

$$\rho^{-1} U \cap f^{-1} V \subset \bigcap_{e^i \succ e^j} \Delta^j.$$

The aggregate of these choices as x varies forms a covering of the compact polyhedron $f^{-1}y$, which admits of a finite subcovering $\rho^{-1} U^k \cap f^{-1} V^k$ $(k = 1, \ldots, n)$, say. Let $W = \bigcap_k V^k$, and choose s minimal such that the star covering of the sth derived of M^i refines $\{U^k\}$. The requirements of the lemma are satisfied.

We can now continue with the construction of α, namely with the inductive choice of s^i and t^i. The sets W of Lemma 10 cover J^i, which is compact. Therefore a finite number of W's cover J^i, and indeed cover their union $\tilde{J}^i$ say, which is an open neighbourhood of J^i in Y. Let s^i be the maximum of the corresponding finite set of s's. If t is large enough, the star covering of $L^{(t)}$ refines the covering by W's of $\tilde{J}^i$. If t is minimal with respect to this property, define $t^i = \max(t+1, t^{i-1})$. We have completed the inductive definition of s^i and t^i.

Let $L_\nu = L^{(s^m)}$. We say that a vertex b_ν of L_ν is of *filtration* i if it is contained in $B^i - B^{i-1}$. If b_ν is of filtration i, let $\alpha(b_\nu)$ be the covering $\gamma(e^i, s^i)/|\operatorname{st} b_\nu|$ of the solid fibre above the star of b_ν. Let

$$\alpha = \bigcup \{\alpha(b_\nu);\, b_\nu \in L_\nu\}.$$

Remark. We have completed the definition of α. However, α is but the first of an infinite sequence of such coverings. The rth covering $\alpha^{(r)}$ is obtained by replacing L and M^i in the construction of α by their rth deriveds. Since X is compact, property (iii) of the polyfibre map ensures that this sequence of coverings is cofinal in the directed set of coverings of X.

Notation

Let $K_\lambda = N^0(\alpha)$, an oriented nerve of α. If β is the star covering of $L^{(s^m)}$, let $L_\nu = L^{(s^m)} = N^0(\beta)$. By construction α refines $f^{-1}\beta$, and there is a unique simplicial approximation $\phi : K_\lambda \to L_\nu$ of f. If T_ν is a set of simplexes of L_ν, let

$$N_\lambda T_\nu = N^0(\alpha/|T_\nu|),$$

the oriented nerve of α cut down to the solid fibre above $|T_\nu|$. We may regard $N_\lambda T_\nu$ as a subcomplex of K_λ, and alternatively define

$$N_\lambda T_\nu = \{\sigma_\lambda ; f(\sup \sigma_\lambda) \cap |T_\nu| \neq \emptyset\}.$$

Since supports are open sets and f is an open map, we deduce that $N_\lambda T_\nu = N_\lambda \bar{T}_\nu$. Therefore we may as well assume from now on that T_ν is a subcomplex of L_ν.

LEMMA 11. *If b is a vertex of L_ν of filtration i, then $N_\lambda b = \phi^{-1} b$, and inclusion induces $H_\lambda N_\lambda b \xrightarrow{\cong} \mathfrak{H} e^i$.*

Proof. First we notice that the only open sets of α which meet $f^{-1} b$ are those of $\alpha(b)$. Property (ii) of a polyfibre map ensures that the support of a simplex in $N^0(\gamma(e^i, s^i))$ meets the fibre above every point of $|\mathrm{st}(e^i, L)|$. Therefore if $\sigma_\lambda \in N^0(\alpha(b))$, then $f(\sup \sigma_\lambda) = |\mathrm{st}(b, L_\nu)|$, and so $\sigma_\lambda \in N_\lambda b$. Therefore

$$N_\lambda b = \phi^{-1} b = N^0(\alpha(b)) \xrightarrow{\cong} N^0(\gamma(e^i, s^i)).$$

But by property (ii) again, these coverings are sufficient to compute the Čech homology groups of $f^{-1} b$ and Fe^i, respectively, so that we have the commutative diagram of isomorphisms

$$\begin{array}{ccc} \check{H}_*(f^{-1} b) & \xrightarrow{\cong} & \check{H}_*(Fe^i) = \mathfrak{H} e^i \\ \downarrow \cong & & \downarrow \cong \\ H_\lambda N_\lambda b & \xrightarrow{\cong} & H_*(N^0(\gamma(e^i, s^i))) \end{array}$$

Remark. We have shown that if T_ν is a vertex, then $N_\lambda T_\nu = \phi^{-1} T_\nu$. If T_ν is not a vertex, however, although $N_\lambda T_\nu$ has the same vertices as $\phi^{-1} T_\nu$, it may be slightly smaller than $\phi^{-1} T_\nu$, so that we can only claim

$$N_\lambda T_\nu \subset \phi^{-1} T_\nu.$$

For example, we may have a simplex whose vertices lie in $\phi^{-1} T_\nu$ but whose support lies above the complement of $|T_\nu|$. And it is this point which is the very crux of the matter. For herein is reflected the main advantage and awkwardness of fibre bundles, whereby two open sets which look nice and horizontal in two neighbouring coordinate systems may cut each other pathologically. It is this point which necessitates the elaborate construction of α and the device of Lemma 13. The crucial property of α is pinpointed in:

LEMMA 12. *If $b^0 b^1 \dots b^q = \tau_\nu \in L_\nu$, the vertices arranged in some order of increasing filtration, then for each $k < l$, $\alpha(b^l)/|\mathrm{st}(b^k b^l, L_\nu)|$ refines*

$\alpha(b^k)/|\operatorname{st}(b^k b^l, L_\nu)|$, *and we can choose a corresponding simplicial approximation* $\zeta^{k,l}: N_\lambda b^l \to N_\lambda b^k$. *The approximations can be chosen to satisfy the associative law* $\zeta^{k,l}\zeta^{l,m} = \zeta^{k,m}$.

Proof. First we define $\zeta^{k,k+1}$ $(k = 0, 1, \ldots, q-1)$. Let the filtrations of b^k, b^{k+1} be j, i respectively. If $i = j$, both $\alpha(b^k)$ and $\alpha(b^{k+1})$ are isomorphic restrictions of $\gamma(e^i, s^i)$, so obviously we choose $\zeta^{k,k+1}$ to be induced by the identity.

If $i > j$, then $e^i \succ e^j$. From the construction of α, we notice that b^{k+1} must be contained in the open simplicial neighbourhood of J^i in $L^{(t^i)}$, and so $b^{k+1} \in |\operatorname{st}(b, L^{(t^i)})|$, some $b \in J^i$. Therefore $|\tau_\nu| \subset |\operatorname{st}(b, L^{(t^i)})|$. Therefore $|\bar{\tau}_\nu| \subset |\operatorname{st}(b, L^{(t)})|$ if $t+1 \leqslant t^i$. (This was the reason for the '$+1$' in the original definition of t^i.) Therefore if $V = |\operatorname{st}(\bar{\tau}_\nu, L_\nu)|$, the open simplicial neighbourhood of $\bar{\tau}_\nu$ in L_ν, then $V \subset |\operatorname{st}(b, L^{(t)})|$. But $|\operatorname{st}(b, L^{(t)})|$ is contained in some W of Lemma 10. Therefore by Lemma 10, $\gamma(e^i, s^i)/V$ refines $\gamma(e^j, s^j)/V$. We can identify the oriented nerves of these coverings with $N_\lambda b^{k+1}$ and $N_\lambda b^k$, respectively, and define an approximation $\zeta^{k,k+1}$ accordingly.

If $k < l$, define $\zeta^{k,l}$ by the composition $\zeta^{k,k+1} \ldots \zeta^{l-1,l}$; clearly the associative law is satisfied. If j, i are now the filtrations of b^k, b^l, respectively, then by induction $\gamma(e^i, s^i)/V$ refines $\gamma(e^j, s^j)/V$ and $\zeta^{k,l}$ is an approximation. Since $V \supset |\operatorname{st}(b^k b^l, L_\nu)|$, we can replace V by $|\operatorname{st}(b^k b^l, L_\nu)|$ in these statements, and we have the lemma.

Definition. Let T_ν be a subcomplex of L_ν. Let τ_ν be a simplex of T_ν, b^0 a vertex of minimum filtration, and $\tau_\nu{}^0$ the opposite face. Let S_ν be T_ν with τ_ν and $\tau_\nu{}^0$ removed. If S_ν is a subcomplex of T_ν, we say that the passage from T_ν to S_ν is a *simple contraction*. (The condition is equivalent to saying τ_ν is a principal simplex of T_ν, and $\tau_\nu{}^0$ is a free face.) We say that T_ν is *simply contractible* to the vertex b, if there is a sequence of simple contractions beginning with T_ν and ending with b (which must therefore be a vertex of minimum filtration in T_ν). As an example, if $e \in L$, and T_ν is the subcomplex of L_ν triangulating $|\operatorname{dual} e|$, then T_ν is simply contractible to $\hat{e}$.

LEMMA 13. *If T_ν is simply contractible to b, then the inclusion $N_\lambda b \subset N_\lambda N_\nu$ is a chain equivalence.*

Proof. The idea is to lift the nibbling away of T_ν by simple contractions to a nibbling away of $N_\lambda T_\nu$. Consequently it is sufficient to show that if the removal of τ_ν and $\tau_\nu{}^0$ from T_ν to S_ν is a simple contraction, then $N_\lambda S_\nu \subset N_\lambda T_\nu$ is a chain equivalence.

If $\sigma_\lambda \in N_\lambda T_\nu - N_\lambda S_\nu$, then $f(\sup \sigma_\lambda)$ meets $|T_\nu|$ but not $|S_\nu|$, and therefore $f(\sup \sigma_\lambda)$, being an open set, meets $|\tau_\nu|$. By excision, it suffices to show that $N_\lambda \tau_\nu \cap N_\lambda S_\nu \subset N_\lambda \tau_\nu$ is a chain equivalence. We prove this by producing a retraction $\rho : N_\lambda \tau_\nu \to N_\lambda b^0$, which is a chain equivalence, and which when restricted to $N_\lambda \tau_\nu \cap N_\lambda S_\nu$ is also a chain equivalence.

Let $\tau_\nu = b^0 b^1 \dots b^q$, with the vertices arranged in some order of increasing filtration. If a is a vertex of $N_\lambda \tau_\nu$, and ϕa is of filtration k, define $\rho a = \zeta^{0,k} a$, where $\zeta^{0,k}$ is given by Lemma 12. Then ρ is a simplicial map $N_\lambda \tau_\nu \to N_\lambda b^0$; for let $a^1 a^2 \dots a^p = \sigma_\lambda \in N_\lambda \tau_\nu$. By Lemma 12,

$$\sup \rho a^i \cap f^{-1}|\tau_\nu| \supset \sup a^i \cap f^{-1}|\tau_\nu|,$$

and so

$$\sup \rho a^1 \cap \dots \cap \sup \rho a^p \cap f^{-1}|\tau_\nu| \supset \sup \sigma_\lambda \cap f^{-1}|\tau_\nu| \neq \emptyset.$$

Therefore $\rho a^1, \dots, \rho a^p$ span a simplex $\rho\sigma_\lambda \in N_\lambda b^0$.

To show that ρ is a chain equivalence we observe that it is contiguous to the identity. For if $\sigma_\lambda \in N_\lambda \tau_\nu$, then

$$\sup \sigma_\lambda \cap \sup \rho\sigma_\lambda \cap f^{-1}|\tau_\nu| \neq \emptyset.$$

Therefore there is a simplex $\chi\sigma_\lambda \in N_\lambda \tau_\nu$ with vertices $a^1, \dots, a^p$, $\rho a^1, \dots, \rho a^p$.

Now let $R_\lambda = N_\lambda \tau_\nu \cap N_\lambda S_\nu$, and let $\bar\rho$ be the restriction of $\bar\rho$ to R_λ. To show that $\bar\rho$ is a chain equivalence is not so easy, because $\bar\rho$ is not in general contiguous to the identity, for the same reason as described in the remark preceding Lemma 12. We have to produce an acyclic carrier Ψ from R_λ to itself, carrying both $\bar\rho$ and the identity.

Suppose $\sigma_\lambda \in R_\lambda$, and let $\tau_\nu{}^1 = \phi\sigma_\lambda$. Then $f(\sup \sigma_\lambda)$ is contained in $|\mathrm{st}\,\tau_\nu{}^1|$, and by hypothesis meets $|\tau_\nu|$ and $|S_\nu|$. Therefore $\tau_\nu{}^1 \in \bar\tau_\nu \cap S_\nu$. Let $\tau_\nu{}^2 = b^0 \tau_\nu{}^1$ if $b^0 \notin \tau_\nu{}^1$, and let $\tau_\nu{}^2 = \tau_\nu{}^1$ if $b^0 \in \tau_\nu{}^1$. Let $U^1 = f^{-1}|\mathrm{st}\,\tau_\nu{}^1|$ and $U^2 = f^{-1}|\mathrm{st}\,\tau_\nu{}^2|$. Suppose that the vertices $a^1, a^2, \dots, a^p$ of σ_λ are labelled so that $\phi a^1, \dots, \phi a^p$ are in the correct ordering of the b's (possibly with repetitions). If $1 \leqslant i < j \leqslant p$, and if k, l are the filtrations of $\phi a^i, \phi a^j$, respectively, let $a^{i,j} = \zeta^{k,l} a^j$. Also let $a^{0,j} = \rho a^j$, and $a^{j,j} = a^j$. We claim that there is a simplex $\psi^0 \sigma_\lambda \in R_\lambda$ spanning the vertices $\{a^{i,j};\ 1 \leqslant i \leqslant j \leqslant p\}$. For, by Lemma 12, if $1 \leqslant i \leqslant j \leqslant p$, then

$$U^1 \cap \sup a^{i,j} \supset U^1 \cap \sup a^{j,j} \supset U^1 \cap \sup \sigma_\lambda = \sup \sigma_\lambda.$$

Therefore the supports of all the vertices concerned meet, and $\psi^0 \sigma_\lambda$ exists in K_λ. Moreover $\sup \psi^0 \sigma_\lambda = \sup \sigma_\lambda$, and since σ_λ satisfies the condition for being in R_λ, so does $\psi^0 \sigma_\lambda$.

We also claim that for each k, $1 \leqslant k \leqslant p$, there is a simplex $\psi^k \sigma_\lambda \in R_\lambda$ spanning the vertices $\{a^{i,j};\ 0 \leqslant i \leqslant k \leqslant j \leqslant p\}$. For if

$$U^3 = U^2 \cap \sup a^{k,k} \cap \sup a^{k,k+1} \cap \dots \cap \sup a^{k,p},$$

then by Lemma 12

$$\begin{aligned} U^3 &\supset f^{-1}|\tau_\nu| \cap \sup a^{k,k} \cap \sup a^{k+1,k+1} \cap \dots \cap \sup a^{p,p} \\ &\supset f^{-1}|\tau_\nu| \cap \sup \sigma_\lambda \\ &\neq \emptyset. \end{aligned}$$

Therefore $a^{k,k}, a^{k,k+1}, \ldots, a^{k,p}$ span a simplex, $\sigma_\lambda{}^k$ say, in $N_\lambda(\phi a^k)$. And if $0 \leqslant i \leqslant k \leqslant j \leqslant p$, then

$$U^2 \cap \sup a^{i,j} \supset U^2 \cap \sup a^{k,j} \supset U^3.$$

Therefore the supports of all the vertices concerned meet, and so $\psi^k \sigma_\lambda$ exists in K_λ. Moreover $U^2 \cap \sup \psi^k \sigma_\lambda = U^3$, and so

$$f(\sup \psi^k \sigma_\lambda) \supset fU^3 = f(U^2 \cap \sup \sigma_\lambda{}^k) = |\operatorname{st} \tau_\nu{}^2|,$$

which meets both $|\tau_\nu|$ and $|S_\nu|$. Therefore $\psi^k \sigma_\lambda$ qualifies for being in R_λ.

We can now define the carrier Ψ by the formula

$$\Psi\sigma_\lambda = \bigcup_{0 \leqslant k \leqslant p} \overline{\psi^k \sigma_\lambda}.$$

It is a carrier because, if $\sigma_\lambda \succ \sigma'_\lambda$, then $\Psi\sigma_\lambda \supset \Psi\sigma'_\lambda$, since each $\psi^k \sigma'_\lambda$ is a face of some $\psi^l \sigma_\lambda$. It is acyclic because $\Psi\sigma_\lambda$ is a cone with vertex $a^{1,p}$. Finally it carries both the identity, because $\psi^0 \sigma_\lambda \succ \sigma_\lambda$, and $\bar{\rho}$, because $\psi^1 \sigma_\lambda \succ \rho\sigma_\lambda$. Lemma 13 is proved.

LEMMA 14. *Inclusions induce an isomorphism* $\tilde{E}^2(f, \alpha, L) \xrightarrow{\cong} H_*(L; \mathfrak{H})$.

Proof. Let $L_\mu = L$. Recall that the computation facing relation between K_λ and L_μ is given by

$$\tilde{\mathfrak{F}} = \{\sigma_\lambda \otimes e_\mu; f(\sup \sigma_\lambda) \cap |\operatorname{dual} e_\mu| \neq \emptyset\}.$$

If T_ν is the subcomplex of L_ν triangulating $|\operatorname{dual} e_\mu|$, the facet $\tilde{\mathfrak{F}}_\lambda e_\mu = N_\lambda T_\nu$. By Lemma 12, $N_\lambda \hat{e}_\mu \subset N_\lambda T_\nu$ is a chain equivalence. Therefore, by Lemma 11, inclusions induce isomorphisms

$$H_\lambda N_\lambda \hat{e}_\mu \xrightarrow{\cong} H_\lambda \tilde{\mathfrak{F}}_\lambda e_\mu \xrightarrow{\cong} \mathfrak{H} e_\mu.$$

The resulting stack isomorphism $H_\lambda \tilde{\mathfrak{F}}_\lambda \xrightarrow{\cong} \mathfrak{H}$ gives the lemma.

Proof of Theorem 4. We shall establish an isomorphism

$$\check{E}(f) \xrightarrow{\cong} \tilde{E}(f, \alpha, L),$$

which together with Theorem 1 and Lemma 14 implies Theorem 4.

Continuing with the notation

$$K_\lambda = N^0(\alpha), \quad L_\mu = L, \quad \text{and} \quad L_\nu = L^{(s^m)} = N^0(\beta),$$

consider the triple facing relation on $K_\lambda \otimes L_\mu \otimes L_\nu$ given by

$$\mathfrak{F} = \{\sigma_\lambda \otimes e_\mu \otimes \tau_\nu; f(\sup \sigma_\nu) \cap |\operatorname{st} \tau_\nu| \neq \emptyset \quad \text{and} \quad |\operatorname{dual} e_\mu| \supset |\tau_\nu|\}.$$

We notice similar properties to those enjoyed by the facing relation in the proof of Theorem 3.

(1) $\mathfrak{F}$ is (μ, ν)-acyclic.

(2) $\mathfrak{F}$ is (λ, μ, ν)-independent.

(3) $\mathfrak{F}$ is (λ, μ, ν)-excisable. For let the acyclic functor $\Gamma : L_\nu \to L_\mu^{\succ}$ carrying $\mathfrak{F}_{\mu\nu}$ be given by $\Gamma\tau_\nu = (e_\mu{}^j, e_\mu{}^j)$, where j is the minimum filtration of the vertices of $\overline{\operatorname{st}\tau_\nu}$. (We observe that we could have used a simpler version of 'excisability' for this particular application, and dispensed with $L_\mu^{\succ}$, for the first member of the pair $(e_\mu{}^i, e_\mu{}^j)$ is irrelevant.) Let

$$C^j = \bigcup_{e^i \succ e^j} (B^i - B^{i-1}).$$

It is possible to show that C^j is the closed subset of Y underlying the subcomplex of L spanned by all vertices of filtration i, $e^i \succ e^j$. Let the facing relation $\mathfrak{F}^{\succ}$ between K_λ and $L_\mu^{\succ}$ be given by

$$\mathfrak{F}^{\succ} = \{\sigma_\lambda \otimes (e_\mu{}^i, e_\mu{}^j); f(\sup \sigma_\lambda) \cap C^j \neq \emptyset\}.$$

We can use Lemma 13 to prove the chain equivalent inclusions

$$N_\lambda \hat{e}_\mu \subset \tilde{\mathfrak{F}}_\lambda e_\mu \subset \mathfrak{F}_\lambda e_\mu \subset \mathfrak{F}_\lambda^{\succ}(e_\mu, e_\mu),$$

and $$\mathfrak{F}_\lambda(e_\mu \otimes \tau_\nu) \subset \mathfrak{F}_\lambda^{\succ} \Gamma\tau_\nu,$$

where $e_\mu \otimes \tau_\nu \in \mathfrak{F}_{\mu\nu}$. Applying Lemmas 7 and 8, we have isomorphisms

$$E_{\nu,\lambda} \xleftarrow[\epsilon_\mu]{\cong} E_{\mu\nu,\lambda} \xrightarrow[\epsilon_\nu]{\cong} E_{\mu,\lambda}.$$

(4) $\mathfrak{F}_{\lambda\nu}$ is none other than the Čech facing relation of f between oriented nerves of α, β, so that by Lemma 4, $E_{\nu,\lambda} \cong \check{E}(f, \alpha, \beta)$.

(5) There is an isomorphism induced by inclusion

$$E^2_{\mu,\lambda} \xrightarrow{\cong} H_*(L; \mathfrak{H}).$$

For the chain equivalence $N_\lambda \hat{e}_\mu \subset \mathfrak{F}_\lambda e_\mu$, mentioned in (3) above, leads, as in Lemma 14, to an isomorphism $H_\lambda \mathfrak{F}_\lambda \xrightarrow{\cong} \mathfrak{H}$ between stacks on L_μ.

Combining the above properties, let π be the composition

$$\check{E}^2(f, \alpha, \beta) = E^2_{\nu,\lambda} \xleftarrow{\cong} E^2_{\mu\nu,\lambda} \xrightarrow{\cong} E^2_{\mu,\lambda} \xrightarrow{\cong} H_*(L; \mathfrak{H}).$$

This is but the first of an infinite sequence of isomorphisms $\{\pi^r\}$, corresponding to the cofinal sequence of pairs of coverings $(\alpha^{(r)}, \beta^{(r)})$. The consequence of such a cofinal sequence is, as in Theorem 3, a canonical isomorphism from the Čech sequence of f

$$\check{E}(f) \xrightarrow{\cong} \check{E}(f, \alpha, \beta).$$

(6) Finally we use the chain equivalent inclusion $\tilde{\mathfrak{F}}_\lambda e_\mu \subset \mathfrak{F}_\lambda e_\mu$, mentioned in (3) above, to deduce a stack isomorphism $H_\lambda \tilde{\mathfrak{F}}_\lambda \to H_\lambda \mathfrak{F}_\lambda$ on L_μ, and hence an isomorphism from the computation sequence

$$\tilde{E}(f, \alpha, L) \xrightarrow{\cong} E_{\mu,\lambda}.$$

Combining, we have the composite isomorphism

$$\check{E}(f) \xrightarrow{\cong} \check{E}(f, \alpha, \beta) = E_{\nu,\lambda} \xleftarrow{\cong} E_{\mu\nu,\lambda} \xrightarrow{\cong} E_{\mu,\lambda} \xleftarrow{\cong} \tilde{E}(f, \alpha, L),$$

which concludes the proof of Theorem 4.

7. Singular theory

In this section we relate the singular and Cech spectral theories by establishing canonical homomorphisms between them. As a corollary we show that the two theories are isomorphic on simplicial maps, and on certain fibre maps. Following Serre (**4**) we use singular cubes to define the singular theory. As mentioned in the introduction, we differ in notation from him, in that our degenerate cubes are degenerate at the *front* rather than at the back, and an n-dimensional cube has its first p coordinates parallel, roughly speaking, to the fibre, and its last $q = n-p$ coordinates parallel to the base.

Singular cubical homology

Let $Q(X) = \Sigma Q_n(X)$ denote the geometric chain complex of singular cubes in a space X. Let $Q^D(X)$ denote the subcomplex of degenerate cubes, i.e. those which are independent of the first coordinate. Let $Q^N(X) = Q(X)/Q^D(X)$, the normalized singular cubical chain complex of X. Then $Q^N(X)$ is a geometric chain complex, whose cells are in one–one correspondence with, and may be identified with, the non-degenerate cubes of X.

The singular spectral sequence of a map, etc.

Let $f: X \to Y$ be a continuous map. We make $A = Q^N(X)$ into a filtered graded differential group as follows. The grading n and differential d are those of $Q^N(X)$. The filtration $\{A_q\}$ of A is the image under the epimorphism $Q(X) \to A$ of a filtration $\{Q(X)_q\}$ of $Q(X)$. Define $Q(X)_q$ to be the subcomplex generated by all cubes u, such that fu depends upon only the last (at most) q coordinates. In other words, if u is n-dimensional then $u \in Q(X)_q$ if $n \leqslant q$, or if $n > q$ and fu is independent of the first $p = n-q$ coordinates. The spectral sequence arising from A is defined to be the *singular spectral sequence* ${}^sE(f)$ of the map f. The sequence runs

$$H_q H_p D \underset{q}{\Longrightarrow} {}^sH_n(X),$$

where ${}^sH_*(X)$ is the singular homology group of X, and D is the bigraded differential group associated with A; the gradings of D are p, q, and the differential d^0 is induced by d and is of degree $(1, 0)$.

Similarly if G is a coefficient group, and R a coefficient ring, we can define the spectral sequence ${}^sE(f; G)$ and the spectral ring ${}^sE^*(f; R)$ from the filtered graded differential groups $A \otimes G$ and $A \pitchfork R$, respectively.

Relation with Serre's theory

The above is a mild generalization of Serre's definition (**4**). He was concerned only with fibre maps having path-connected base and fibre, and as a result was able to confine his attention to the subcomplex A^0 of A, comprised of those singular cubes having all their vertices at a base point x^0 in X. The advantage of doing this was that it simplified the identification of the E^2 term with the homologies of base and fibre. Since we are concerned with arbitrary maps, which may have different fibres above different points, we have to allow ourselves singular cubes within every fibre, and so must free the vertices of the cubes from the base point.

It is possible to show that the above definition is a true generalization of Serre's, by proving that in the case of a fibre space the inclusion $j : A^0 \to A$ induces an isomorphism of spectral sequences. For one can define a retraction $k : A \to A^0$ and a homotopy operator $h : jk \simeq 1$, and apply Lemma 2. To define h and k, one uses the homotopy extension and homotopy lifting theorems to lift a similar construction on the singular complex of the base space Y, building inductively on dimension, and remembering to be careful on degenerate cubes (in a manner not unlike that of (4) Chapitre II, Lemmes 4, 5). The construction is straightforward but laborious, and is left to the reader.

THEOREM 5. *Let $f : X \to Y$ be a continuous map, G a coefficient group, and R a coefficient ring. There is a canonical homomorphism*

$$\Upsilon : {}^sE(f; G) \to \check{E}(f; G)$$

from the singular spectral sequence of f to the Čech semi-spectral sequence, and a canonical homomorphism $\Upsilon^* : \check{E}^*(f; R) \to {}^sE^*(f; R)$ from the Čech spectral ring to the singular spectral ring.

COROLLARY 5.1. $\Upsilon : {}^sE \to \check{E}$ *and* $\Upsilon^* : \check{E}^* \to {}^sE^*$ *are natural transformations between functors.*

COROLLARY 5.2. *If $f : Y \to Y$ is the identity, then Υ reduces to the natural*† *transformation from the singular homology group of Y to the Čech homology group, and Υ^* similarly.*

Small cubes

The proof of Theorem 5 will employ a facing relation which fits singular cubes inside Čech supports. Let α be a covering of X; we say that a singular cube of X is α-small if its image lies in some set of α. Denote, as in (8), by $Q(X, \alpha)$ the subcomplex of $Q(X)$ generated by α-small cubes. Let $Q^N(X, \alpha)$ be the induced subcomplex of $Q^N(X)$. Then $A(\alpha) = Q^N(X, \alpha)$ inherits the filtered graded differential structure from A, and so gives rise to a spectral sequence which we denote by ${}^sE(f, \alpha)$.

LEMMA 15. *Inclusion induces an isomorphism* ${}^sE(f, \alpha) \xrightarrow{\cong} {}^sE(f)$, *etc.*

† ((8) Corollary 2.7.)

Proof. The pattern of proof is similar to that of the classical result, that the singular homology groups of a space can be calculated using only small singular simplexes. We have to adapt the proof to cubes and to spectral sequences.

First we give a formula for chopping an n-dimensional cube into 3^n smaller cubes. Let $u : I^n \to X$ be an n-dimensional singular cube of X. Let

$$(\chi_i^{(0)} u)(t_1, \ldots, t_n) = u\left(t_1, \ldots, t_{i-1}, \frac{t_i}{3}, t_{i+1}, \ldots, t_n\right),$$

$$(\chi_i^{(1)} u)(t_1, \ldots, t_n) = u\left(t_1, \ldots, t_{i-1}, \frac{1+t_i}{3}, t_{i+1}, \ldots, t_n\right),$$

$$(\chi_i^{(2)} u)(t_1, \ldots, t_n) = u\left(t_1, \ldots, t_{i-1}, \frac{3-t_i}{3}, t_{i+1}, \ldots, t_n\right),$$

$$(h_i u)(t_1, \ldots, t_{n+1}) = u\left(t_1, \ldots, t_{i-1}, t_i + \frac{(1-2t_i)\,t_{i+1}}{3}, t_{i+2}, \ldots, t_{n+1}\right),$$

for $i = 1, 2, \ldots, n$. Let

$$\chi_i = \chi_i^{(0)} + \chi_i^{(1)} - \chi_i^{(2)},$$

$$\chi = \chi_n \chi_{n-1} \cdots \chi_1,$$

$$h = \sum_{j=1}^{n} (-)^{j-1} h_j \chi_{j-1} \chi_{j-2} \cdots \chi_1.$$

Then χ is a chain map $Q(X) \to Q(X)$, and h is a homotopy operator $Q_n(X) \to Q_{n+1}(X)$, such that $\partial h + h\partial = 1 - \chi$. Moreover the subcomplex of degenerate cubes is stable under χ and h, so the same formula is induced in the normalized complex. If u is such that fu is independent of its first p coordinates, then χu and hu are sums of like cubes. Therefore, passing to the filtered group (and omitting the grading), we see that $\chi(A_q) \subset A_q$ and $h(A_q) \subset A_{q+1}$. The result of chopping-up is that if we apply the operator χ enough times, then any cube is chopped into α-small cubes, or more precisely: given a covering α, and a chain $x \in A_q$, then $\chi^s x \in A_q(\alpha)$ provided s is sufficiently large. If $\psi = 1 + \chi + \chi^2 + \ldots + \chi^{s-1}$, then ψ is a chain map $A \to A$, and

$$1 = \chi^s + h\psi d + dh\psi. \tag{1}$$

To prove the lemma, it is sufficient to show that inclusion induces an isomorphism between the E^2 terms, or in other words that

$$\frac{C_q^2(\alpha)}{C_{q-1}^1(\alpha) + B_q^1(\alpha)} \xrightarrow{\cong} \frac{C_q^2}{C_{q-1}^1 + B_q^1},$$

where B_q^r, C_q^r are the customary groups of boundaries and cycles used in constructing the spectral sequence. The isomorphism holds provided

$$C_q^2 \subset C_q^2(\alpha) + (C_{q-1}^1 + B_q^1), \tag{2}$$

$$C_q^2(\alpha) \cap (C_{q-1}^1 + B_q^1) \subset C_{q-1}^1(\alpha) + B_q^1(\alpha). \tag{3}$$

Let $x \in C_q^2$, the left-hand side of (2). Choose s sufficiently large so that $\chi^s x \in C_q^2(\alpha)$. Applying (1) we have

$$x = \chi^s x + h\psi dx + dh\psi x.$$

But $hd\psi x \in C_{q-1}^1$, because $\psi dx \in A_{q-2}$, so $h\psi dx \in A_{q-1}$, and

$$dhd\psi x = d(1-\chi^s)\, x \in A_{q-2}.$$

Meanwhile $dh\psi x \in B_q^1$, because $h\psi x \in A_{q+1}$, and $dh\psi x = (1-\chi^s)\,x - h\psi dx \in A_q$. Therefore x lies in the right-hand side of (2), and we have proved (2).

Now let $x \in C_q^2(\alpha) \cap (C_{q-1}^1 + B_q^1)$, the left-hand side of (3). Then $x = y + dz$, where $y \in C_{q-1}^1$ and $z \in C_{q+1}^1$. Choose s large enough so that $\chi^s y \in C_{q-1}^1(\alpha)$. Then by (1)

$$y = \chi^s y + h\psi dy + dh\psi y.$$

But $h\psi dy$ is also in $C_{q-1}^1(\alpha)$, because $dy = dx$, and $h\psi dy \in h\psi A_{q-2}(\alpha) \subset A_{q-1}(\alpha)$, and $d(h\psi dy) = d(1-\chi^s)\, y \in A_q(\alpha)$. Therefore $x = y' + dz'$, where

$$y' = \chi^s y + h\psi dy \in C_{q-1}^1(\alpha), \quad \text{and} \quad z' = z + h\psi y \in C_{q+1}^1.$$

To establish (3), it remains to show that $dz' \in B_q^1(\alpha)$. Choose t sufficiently large so that $\chi^t z' \in C_{q+1}^1(\alpha)$. Therefore $dz' = d\chi^s z' + dh\psi dz'$, by (1). But $dz' = x - y' \in A_q(\alpha)$, and so $h\psi dz' \in A_{q+1}(\alpha)$, with $d(h\psi dz') = (1-\chi^s)\, dz' \in A_q(\alpha)$. Therefore $h\psi dz' \in C_{q+1}^1(\alpha)$, and consequently $dz' = d(\chi^s z' + h\psi dz') \in B_q^1(\alpha)$, as desired. We have established (3), and completed the proof of Lemma 15.

The multiple facing relation

In the proof of Theorem 5 we shall use a triple facing relation involving three gradings and one filtration. Let $(f, \alpha, \beta) \in \mathfrak{M}_{\text{cov}}$; in other words, $f: X \to Y$ is a continuous map, and α, β are coverings of X, Y such that α refines $f^{-1}\beta$. Let

$K_\lambda = N(\alpha)$, the graded differential group, graded by p_λ, with differential ∂_λ;

$L_\nu = N(\beta)$, the graded differential group, graded by q_ν, with differential ∂_ν;

$K_{\xi\eta} = Q^N(X, \alpha)$, the filtered graded differential group, filtered by q_η, graded by $n_{\xi\eta}$, with differential $\partial_{\xi\eta}$.

Then $K_\lambda \otimes L_\nu \otimes K_{\xi\eta}$ is a filtered trigraded tridifferential group, filtered by q_η, graded by $p_\lambda, q_\nu, n_{\xi\eta}$, and with (skew-commutative) differentials $d_\lambda, d_\nu, d_{\xi\eta}$. Let $\mathfrak{F}$ be the facing relation on $K_\lambda \otimes L_\nu \otimes K_{\xi\eta}$ given by

$$\mathfrak{F} = \{\sigma_\lambda \otimes \tau_\nu \otimes \sigma_{\xi\eta};\ \sup \sigma_\lambda \supset \operatorname{im} \sigma_{\xi\eta} \text{ and } \sup \tau_\nu \supset \operatorname{im} f\sigma_{\xi\eta}\}. \quad \text{(See Fig. 4.)}$$

Clearly the facing condition is satisfied, since supports expand and images shrink when passing to faces. $\mathfrak{F}$ gives rise to a filtered trigraded tridifferential subgroup $A_{\lambda\nu\xi\eta}$ of $K_\lambda \otimes L_\nu \otimes K_{\xi\eta}$. Let $D_{\lambda\nu\xi\eta}$ be the associated quadrigraded tridifferential group, graded by $p_\lambda, q_\nu, p_\xi, q_\eta$, with induced differentials d_λ, d_ν, d_ξ. We can form various spectral sequences from $A_{\lambda\nu\xi\eta}$, subject to the condition that we must not have both ξ in the filtering degree and η in the complementary degree.

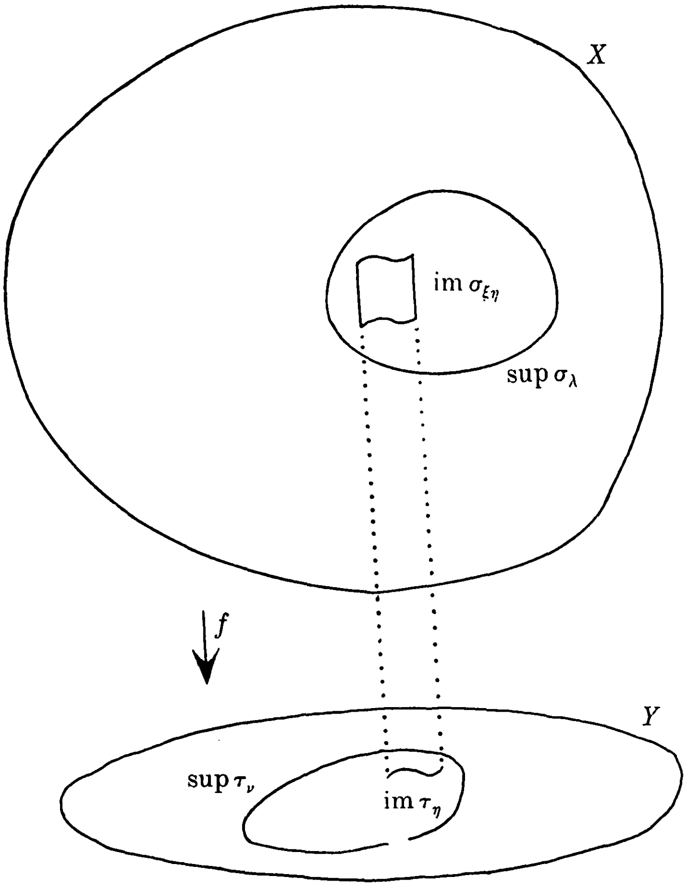

FIG. 4

By applying Lemmas 5 and 7 we shall now deduce the isomorphisms

$$E_{\nu\eta,\lambda\xi} \xrightarrow[\epsilon_\lambda]{\cong} E_{\nu\eta,\xi} \xrightarrow[\epsilon_\nu]{\cong} E_{\eta,\xi}. \tag{1}$$

Notice that the proofs of the lemmas remain valid in spite of the presence of a filtration instead of one of the gradings, because they are performed over the associated graded groups. The requirement for the first isomorphism is that $\mathfrak{Y}$ should be $(\lambda, \nu\xi\eta)$-acyclic, which is true because we have arranged for the singular cubes to be α-small, so that the facets concerned are Čech cones. The requirement for the second isomorphism is given by

LEMMA 16. *$\mathfrak{F}$ is (ν,η)-acyclic and (ξ,ν,η)-independent.*

Proof. The first thing to do is to define the 'facets' $\mathfrak{F}_\nu\tau_\eta, \mathfrak{F}_\xi\tau_\eta, \mathfrak{F}_{\nu\xi}\tau_\eta$; for as yet we have not defined any facets in which the suffixes ξ and η do not occur together. Let $L_\eta = Q^N(Y,\beta)$. For each $\tau_\eta \in L_\eta$, the facets $\mathfrak{F}_\nu\tau_\eta, \mathfrak{F}_\xi\tau_\eta, \mathfrak{F}_{\nu\xi}\tau_\eta$, that we shall construct, will be subgroups of $L_\nu, K_{\xi\eta}, D_{\nu\xi\eta}$, generated by cells and closed with respect to the differentials $d_\nu, d_\xi, d_{\nu\xi}$, respectively, and such that if we take direct sums over all $\tau_\eta \in L_\eta$, then

$$\Sigma\mathfrak{F}_\nu\tau_\eta \otimes \tau_\eta = D_{\nu\eta}, \quad \Sigma\mathfrak{F}_\xi\tau_\eta = K_{\xi\eta}, \quad \Sigma\mathfrak{F}_{\nu\xi}\tau_\eta = D_{\nu\xi\eta}.$$

It is these properties which entitle us to call them facets, and make the statement of the lemma meaningful. It is also these properties of facets which are used in the proof of, and validate the application of, Lemma 7.

Let $\mathfrak{F}_{\nu\eta}$ be the Čech-singular facing relation of (Y,β), as defined in ((8) Example 5); it is the facing relation between L_ν and L_η given by

$$\mathfrak{F}_{\nu\eta} = \{\tau_\nu \otimes \tau_\eta;\ \sup\tau_\nu \supset \operatorname{im}\tau_\eta\}.$$

The facets $\mathfrak{F}_\nu\tau_\eta$ are therefore defined, and being Čech cones are acyclic. Hence $\mathfrak{F}$ is (ν,η)-acyclic, as desired.

If $\sigma_{\xi\eta} \in K_{\xi\eta}$ is a $(p+q)$-dimensional non-degenerate cube in X of filtration precisely q, let $\mathfrak{f}\sigma_{\xi\eta}$ be the q-dimensional cube in Y defined by

$$(\mathfrak{f}\sigma_{\xi\eta})(t_1,\ldots,t_q) = (f\sigma_{\xi\eta})(s_1,\ldots,s_p,t_1,\ldots,t_q), \text{ for any } s_1,\ldots,s_p.$$

By definition of the filtration q, $\mathfrak{f}\sigma_{\xi\eta}$ is a non-degenerate cube, independent of the arbitrary choices of $s_1,\ldots,s_p$. It is β-small, because $\sigma_{\xi\eta}$ is α-small and α refines $f^{-1}\beta$. Therefore $\mathfrak{f}\sigma_{\xi\eta} \in L_\eta$.

We digress for a moment to glance at the resulting commutative square

$$\begin{array}{ccc} I^p \times I^q & \xrightarrow{\sigma_{\xi\eta}} & X \\ \downarrow g & & \downarrow f, \\ I^q & \xrightarrow[\mathfrak{f}\sigma_{\xi\eta}]{} & Y \end{array}$$

where I^p, I^q are standard Euclidean cubes, and g is the projection. For this square is the real structure implicit in a singular cube of filtration q. The left-hand side g may be regarded as a prototype continuous map, just as I^p is a prototype topological space; and just as we obtain the singular homology of a space by considering all maps of the prototype cube into the space, so we obtain the singular spectral theory of f by considering all maps of the prototype g into f, or, in other words, all squares like the above.

We now return to the business of defining the ξ-facets. Let

$$\mathfrak{F}_\xi\tau_\eta = \{\sigma_{\xi\eta};\ \mathfrak{f}\sigma_{\xi\eta} = \tau_\eta\}.$$

This is closed with respect to d_ξ, because d_ξ is, in effect, the boundary with respect to the fibre I^p only, and, when applied to $\sigma_{\xi\eta}$, yields a sum of cubes lying above the same τ_η as $\sigma_{\xi\eta}$. Summing, $\Sigma\mathfrak{F}_\xi\tau_\eta = K_{\xi\eta}$, as was needed. Finally define

$$\mathfrak{F}_{\nu\xi}\tau_\eta = \{\tau_\nu\otimes\sigma_{\xi\eta};\ \sup\tau_\nu\supset\tau_\eta = \mathfrak{f}\sigma_{\xi\eta}\}.$$

Again we see that $\mathfrak{F}_{\nu\xi}\tau_\eta$ is closed with respect to d_ν and d_ξ, and $\Sigma\mathfrak{F}_{\nu\xi}\tau_\eta = D_{\nu\xi\eta}$. Moreover

$$\mathfrak{F}_{\nu\xi}\tau_\eta = \mathfrak{F}_\nu\tau_\eta\otimes\mathfrak{F}_\xi\tau_\eta,$$

from the independence of the conditions $\sup\tau_\nu\supset\tau_\eta$ and $\tau_\eta = \mathfrak{f}\sigma_{\xi\eta}$. Therefore $\mathfrak{F}$ is (ν,ξ,η)-independent, which is the same as (ξ,ν,η)-independent.

Proof of Theorem 5. The facing relation $\mathfrak{F}$ has the following properties:

(1) The $\lambda\nu$-augmentation induces an isomorphism $E_{\nu\eta,\lambda\xi}\xrightarrow{\cong}E_{\eta,\xi}$, as we have already proved.

(2) The $\xi\eta$-augmentation induces a homomorphism $E_{\nu\eta,\lambda\xi}\to E_{\nu,\lambda}$.

(3) The spectral sequence $E_{\eta,\xi}$ is none other than the singular sequence ${}^sE(f,\alpha,\beta)$, and by Lemma 15 the inclusion induces an isomorphism ${}^sE(f,\alpha,\beta)\xrightarrow{\cong}{}^sE(f)$.

(4) $\mathfrak{F}_{\lambda\nu}$ is the Čech facing relation, so that $E_{\nu,\lambda} = \check{E}(f,\alpha,\beta)$.

(5) The facing relation $\mathfrak{F}$ is functorial on $\mathfrak{M}_{\text{cov}}$. For let

$$(\phi,\psi,\phi_\alpha,\psi_\beta):(f,\alpha,\beta)\to(f',\alpha',\beta')$$

be a map of $\mathfrak{M}_{\text{cov}}$, as described in § 2. If

$$\sigma_\lambda\otimes\tau_\nu\otimes\sigma_{\xi\eta}\in\mathfrak{F}(f,\alpha,\beta),\quad\text{then}\quad\phi_\alpha\sigma_\lambda\otimes\psi_\beta\tau_\nu\otimes\phi\sigma_{\xi\eta}\in\mathfrak{F}(f',\alpha',\beta'),$$

because $\qquad\sup\phi_\alpha\sigma_\lambda\supset\phi(\sup\sigma_\lambda)\supset\phi(\operatorname{im}\sigma_{\xi\eta}) = \operatorname{im}\phi\sigma_{\xi\eta},$

and $\qquad\sup\psi_\beta\tau_\nu\supset\psi(\sup\tau_\nu)\supset\psi(\operatorname{im}f\sigma_{\xi\eta}) = \operatorname{im}\psi f\sigma_{\xi\eta} = \operatorname{im}f'\phi\sigma_{\xi\eta}.$

Combining the results of (1) to (4), we define Υ to be the composite

$$\underbrace{{}^sE(f)\xleftarrow{\cong}{}^sE(f,\alpha,\beta) = E_{\eta,\xi}\xleftarrow{\cong}E_{\nu\eta,\lambda\xi}\xrightarrow{\cong}E_{\nu,\lambda} = \check{E}(f,\alpha,\beta)}_{\Upsilon}.$$

Since $\mathfrak{F}$ is functorial, and since the left isomorphism is induced by inclusion, Υ is a natural transformation on $\mathfrak{M}_{\text{cov}}$. Taking inverse limits, we see that $\Upsilon:{}^sE\to\check{E}$ is a natural transformation between the functors ${}^sE,\check{E}:\mathfrak{M}\to\mathfrak{E}$. This completes the proof of Theorem 5 and Corollary 5.1, subject to the observation that the proof over an arbitrary coefficient group G, and the dual proof over an arbitrary ring R, are similar.

Proof of Corollary 5.2. If $f: Y\to Y$ is the identity, we may restrict ourselves to the cofinal system of coverings in which $\alpha = \beta$. Then all the spectral sequences in the above row collapse, and we have the commutative diagrams

$$\begin{array}{ccccccc}
{}^sH_*(Y) & \xleftarrow{\cong} & H_\eta L_\eta & \xleftarrow{\cong} & H_{\nu\eta} D_{\nu\eta} & \longrightarrow & H_\nu L_\nu = H_*(N(\beta)) \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\
{}^sE^2(f) & \xleftarrow{\cong} & E^2_{\eta,\xi} & \xleftarrow{\cong} & E^2_{\nu\eta,\lambda\xi} & \longrightarrow & E^2_{\nu,\lambda} = E^2(f,\beta,\beta)
\end{array},$$

which give in the limit

$$\begin{array}{ccc}
{}^sH_*(Y) & \longrightarrow & \check{H}_*(Y) \\
\downarrow\cong & & \downarrow\cong \\
{}^sE^2(f) & \xrightarrow{\Upsilon} & \check{E}^2(f)
\end{array}.$$

The top homomorphism arises from the Čech-singular facing relation $\mathfrak{F}_{\nu\eta}$ on Y, and so is the natural transformation from singular to Čech homology ((8) Corollary 2.7).

Fibre spaces

In the case of a fibre space, we can identify ${}^sE^*(f; R)$ with Serre's spectral ring, and use Theorem 2 to identify $\check{E}^*(f; R)$ with Leray's. Also the simple structure of the E_2 terms enables us to impose sufficient conditions on f for Υ^* to be an isomorphism. The conditions in Theorem 6 below are merely those which permit of an immediate application of the comparison theorem for spectral sequences (7). We cannot do the same for homology, because the comparison theorem fails for semi-spectral sequences. No doubt the conditions of Theorem 6 could be weakened. In particular it would be pleasant to replace condition (i) by something which ensured the correct E_2 term for the Leray sequence, and which included the loop-space fibring of a nice space.

THEOREM 6. *If $f: X \to Y$ is a fibre map with connected fibre and base, Υ^* is a natural transformation from Leray's spectral ring to Serre's spectral ring. Υ^* is an isomorphism under the following conditions:*

(i) *The fibre map f is locally trivial.*

(ii) *The fundamental group of Y acts trivially on the Čech and singular cohomology rings of the fibre F.*

(iii) *The Čech and singular cohomology groups of Y are finitely generated in each dimension.*

(iv) *Two of the spaces X, Y, F have their Čech and singular cohomology rings canonically isomorphic.* (*Therefore the third has also.*)

We shall conclude the paper by proving that Υ, Υ^* are isomorphisms if f underlies a simplicial map. A preliminary lemma is necessary.

Definition. Suppose we are given a map $f: X \to Y$, and a subspace $Y^1 \subset Y$. Let $X^1 = f^{-1}(Y^1)$. Denote by f/Y^1 the map $X^1 \to Y^1$ induced by f. We say that f/Y^1 is a *deformation retract* of f if there exist deformation retractions $a: X \times I \to X$ of X onto X^1 keeping X^1 fixed, and $b: Y \times I \to Y$ of Y onto Y^1 keeping Y^1 fixed, such that $fa = b(f \times 1)$. Recall also the notation that if α is a covering of X, then α/Y^1 is the induced covering of X^1.

LEMMA 17. *If f/Y^1 is a deformation retract of f, then the inclusion induces an isomorphism* ${}^sE(f/Y^1, \alpha/Y^1) \xrightarrow{\cong} {}^sE(f, \alpha)$, *etc.*

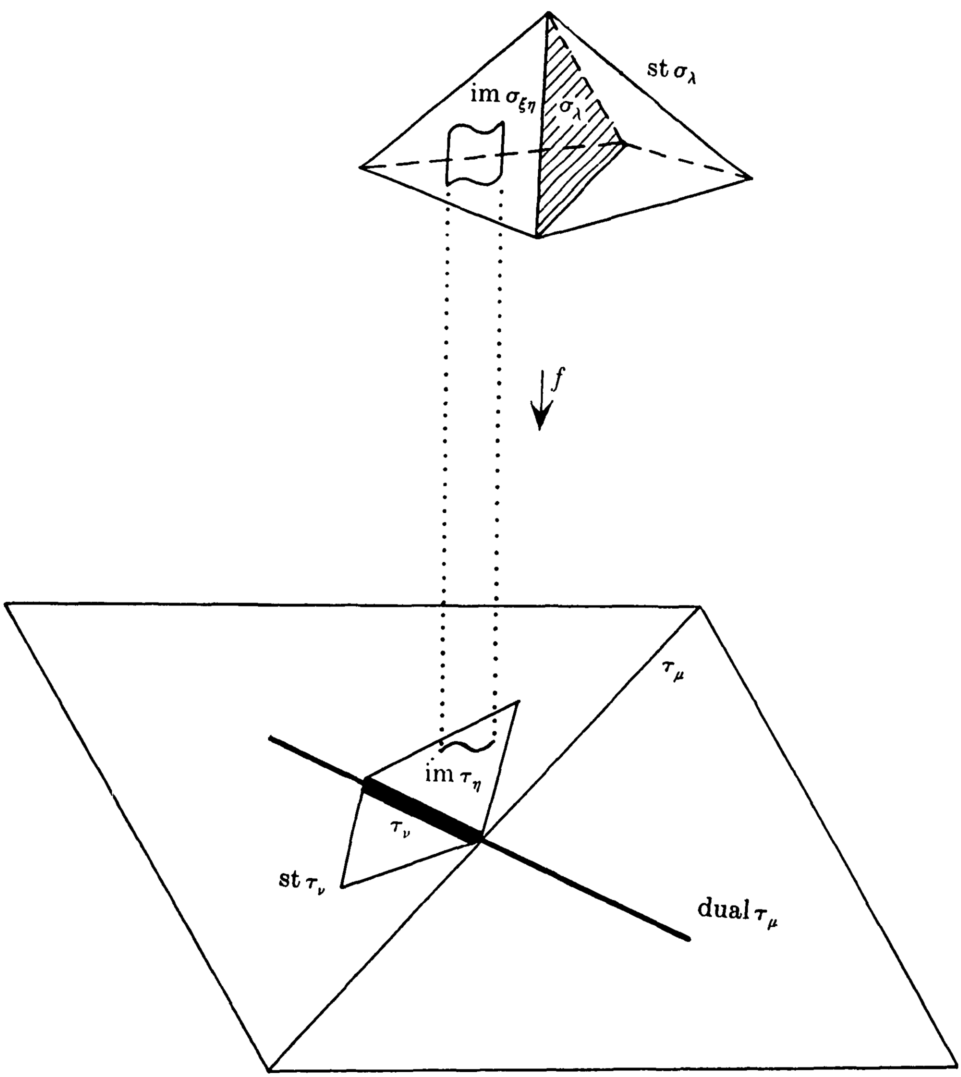

FIG. 5

Proof. Let $j : Q^N(X^1) \to Q^N(X)$ be the inclusion, and $k : Q^N(X) \to Q^N(X^1)$ the homomorphism induced by the retraction $\rho : X \to X^1$, where $\rho x = a(x, 1)$. Then $kj = 1$, because a is fixed on X^1. Let $h : Q_n(X) \to Q_{n+1}(X)$ be defined by $hu = (-)^{n+1} v$, where

$$v(t_1, \ldots, t_{n+1}) = a(u(t_1, \ldots, t_n), t_{n+1}).$$

Notice that h preserves degeneracies, and so induces a homotopy operator on $Q^N(X)$ such that $\partial h + h\partial = 1 - jk$. Passing to the filtration $\{A_q\}$ of $Q^N(X)$, we see that $kA_q \subset A_q$ and $hA_q \subset A_{q+1}$. Therefore, by Lemma 2, k induces an isomorphism ${}^sE(f) \xrightarrow{\cong} {}^sE(f/Y^1)$, and j the inverse isomorphism. Lemma 17 follows from Lemma 15, and the commutative diagram induced by inclusions

$$\begin{array}{ccc} {}^sE(f/Y^1, \alpha/Y^1) & \longrightarrow & {}^sE(f, \alpha) \\ \downarrow \cong & & \downarrow \cong \\ {}^sE(f/Y^1) & \xrightarrow{\cong} & {}^sE(f) \end{array} .$$

THEOREM 7. *All the spectral theories of a simplicial map are isomorphic.*

Proof. We have already shown in Theorems 2 and 3 that the simplicial, Čech, and Leray theories agree. It remains to bring the singular theory into the fold. For this we use a quadruple facing relation, which is a combination of the two triple facing relations used in Theorems 3 and 5.

Suppose $f^{(0)} : K \to L$ is the simplicial map, and $f : X \to Y$ the underlying continuous map. Let $K_\lambda = K^{(2)}$, $L_\mu = L$, $L_\nu = L^{(2)}$, and $K_{\xi\eta} = Q^N(X, \alpha)$, where α is the star covering of $K^{(2)}$. Let $\mathfrak{F}$ be the facing relation on $K_\lambda \otimes L_\mu \otimes L_\nu \otimes K_{\xi\eta}$ given by

$$\mathfrak{F} = \{\sigma_\lambda \otimes \tau_\mu \otimes \tau_\nu \otimes \sigma_{\xi\eta};\ |\operatorname{st} \sigma_\lambda| \supset \operatorname{im} \sigma_{\xi\eta},$$
$$|\operatorname{st} \tau_\nu| \supset \operatorname{im} f\sigma_{\xi\eta}, \quad \text{and} \quad |\operatorname{dual} \tau_\mu| \supset |\tau_\nu|\}.$$

(See Fig. 5.)

Consider the diagram

$$\begin{array}{ccccccc} E_{\mu\eta,\lambda\xi} & \xleftarrow[\epsilon_\nu{}^1]{} & E_{\mu\nu\eta,\lambda\xi} & \xrightarrow[\epsilon_\mu{}^1]{} & E_{\nu\eta,\lambda\xi} & \xrightarrow{\cong} & {}^sE(f) \\ \downarrow \epsilon_{\xi\eta}^1 & & \downarrow \epsilon_{\xi\eta}^2 & & \downarrow \epsilon_{\xi\eta}^3 & & \downarrow \Upsilon \\ E_{\mu,\lambda} & \xleftarrow[\epsilon_\nu{}^2]{\cong} & E_{\mu\nu,\lambda} & \xrightarrow[\epsilon_\mu{}^2]{\cong} & E_{\nu,\lambda} & \xleftarrow{\cong} & \check{E}(f) \end{array} .$$

The bottom row of isomorphisms was proved in Theorem 3, and the top right isomorphism in Theorem 5. The right-hand square is commutative, since it is the definition of Υ. The other two squares are commutative, because they are induced by augmentations. To prove that Υ is an isomorphism, it is sufficient to prove that $\epsilon_\mu{}^1$, $\epsilon_\nu{}^1$, and $\epsilon_{\xi\eta}^1$ are isomorphisms.

Of course it would be simpler if we could prove that $\epsilon_{\xi\eta}^3$ was an isomorphism, but we cannot do this directly, because in the facing relation $\mathfrak{F}_{\lambda\nu\xi\eta}$ we have not fed in the essential fact that $f^{(2)}$ is the second derived of a simplicial map.

We deduce that $\epsilon_\mu{}^1$ is an isomorphism by Lemma 7, because $\mathfrak{F}$ is $(\mu, \nu\eta)$-acyclic and $(\lambda\xi, \mu, \nu\eta)$-independent in the sense of Lemma 16. Similarly $\epsilon_\nu{}^1$ is an isomorphism, because $\mathfrak{F}$ is $(\nu, \mu\eta)$-acyclic and $(\lambda\xi, \mu, \nu\eta)$-independent. It is intriguing to notice how the presence of $\xi\eta$ lightens the task of proving $\epsilon_\nu{}^1$ an isomorphism, compared with $\epsilon_\nu{}^2$; for the proof that $\epsilon_\nu{}^2$ was an isomorphism in Theorem 3 required all the elaborate construction of (λ, μ, ν)-excisability, which is apparently unavoidable. The third isomorphism, $\epsilon_{\xi\eta}^1$, is given by the following lemma, and completes the proof of Theorem 7.

LEMMA 18. *The $\xi\eta$-augmentation induces an isomorphism*

$$\epsilon_{\xi\eta} : E_{\mu\eta,\lambda\xi} \xrightarrow{\cong} E_{\mu,\lambda}.$$

Proof. We use Lemma 17, and a variation of Lemma 6. Fix $\tau_\mu \in L_\mu$ for the moment. Let Y^1 denote the point set underlying the open simplicial neighbourhood of $|\operatorname{dual} \tau_\mu|$ in L_ν, and let $X^1 = f^{-1}(Y^1)$. The facet $\mathfrak{F}_{\lambda\xi\eta}\tau_\mu$ is the same as the facing relation $\mathfrak{F}^1$ on $K_\lambda \otimes K_{\xi\eta}$ given by

$$\mathfrak{F}^1 = \{\sigma_\lambda \otimes \sigma_{\xi\eta};\ |\operatorname{st} \sigma_\lambda| \cap X^1 \supset \operatorname{im} \sigma_{\xi\eta}\}.$$

$\mathfrak{F}^1$ gives rise to the filtered bigraded bidifferential group $A_{\lambda\xi\eta}^1 = \mathfrak{F}_{\lambda\xi\eta}\tau_\mu$, with associated trigraded bidifferential group $D_{\lambda\xi\eta}^1$, say.

Let $Y^0 = \hat{\tau}_\mu$, the barycentre of τ_μ, and let $X^0 = f^{-1}(Y^0)$. Let $\mathfrak{F}^0$ be the facing relation on $K_\lambda \otimes K_{\xi\eta}$ given by

$$\mathfrak{F}^0 = \{\sigma_\lambda \otimes \sigma_{\xi\eta};\ |\operatorname{st} \sigma_\lambda| \cap X^0 \supset \operatorname{im} \sigma_{\xi\eta}\}.$$

The resulting group $A_{\lambda\xi\eta}^0$ is a subgroup of, and has a similar structure to, $A_{\lambda\xi\eta}^1$. We list some of the properties of $\mathfrak{F}^0$ and $\mathfrak{F}^1$.

(1) If we forget the filtration, $\mathfrak{F}^0$ is the simplicial-singular facing relation on the polyhedron X^0 ((8) Example (2)), and is consequently both $(\lambda, \xi\eta)$-acyclic and $(\xi\eta, \lambda)$-acyclic. Therefore

$$H_{\xi\eta} A_{\xi\eta}^0 \xleftarrow[\epsilon_\lambda]{\cong} H_{\lambda\xi\eta} A_{\lambda\xi\eta}^0 \xrightarrow[\epsilon_{\xi\eta}]{\cong} H_\lambda A_\lambda{}^0.$$

(2) $\mathfrak{F}^1$ is $(\lambda, \xi\eta)$-acyclic, so that

$$E_{\eta,\lambda\xi} A_{\lambda\xi\eta}^1 \xrightarrow[\epsilon_\lambda]{\cong} E_{\eta,\xi} A_{\xi\eta}^1,$$

and in particular

$$H_{\lambda\xi\eta} A_{\lambda\xi\eta}^1 \xrightarrow[\epsilon_\lambda]{\cong} H_{\xi\eta} A_{\xi\eta}^1.$$

(3) X^0 is a deformation retract of $\overline{X^1}$, and since $A_\lambda{}^0, A_\lambda{}^1$ are the subcomplexes of K_λ triangulating $X^0, \overline{X^1}$, respectively, the inclusion induces an isomorphism $H_\lambda A_\lambda{}^0 \xrightarrow{\cong} H_\lambda A_\lambda{}^1$.

(4) The map f/Y^0 is a deformation retract of f/Y^1, and so, by Lemma 17, the inclusion induces an isomorphism

$$E_{\eta,\xi} A^0_{\xi\eta} = {}^s E(f/Y^0, \alpha/Y^0) \xrightarrow{\cong} {}^s E(f/Y^1, \alpha/Y^1) = E_{\eta,\xi} A^1_{\xi\eta},$$

and in particular

$$H_{\xi\eta} A^0_{\xi\eta} \xrightarrow{\cong} H_{\xi\eta} A^1_{\xi\eta}.$$

(5) ${}^s E(f/Y^0, \alpha/Y^0)$ collapses onto the axis $q_\eta = 0$, since Y^0 is a point.

(6) From (2), (4), and (5) we deduce that $E_{\eta,\lambda\xi} A_{\lambda\xi\eta}^1$ also collapses, so that

$$H_\eta H_{\lambda\xi} D^1_{\lambda\xi\eta} \xrightarrow{\cong} H_{\lambda\xi\eta} A^1_{\lambda\xi\eta}.$$

There is a commutative diagram, whose horizontal homomorphisms are induced by augmentation, and vertical homomorphisms by inclusion:

$$\begin{array}{ccccc} H_{\xi\eta} A^0_{\xi\eta} & \xleftarrow[\epsilon_\lambda]{\cong} & H_{\lambda\xi\eta} A^0_{\lambda\xi\eta} & \xrightarrow[\epsilon_{\xi\eta}]{\cong} & H_\lambda A_\lambda{}^0 \\ \downarrow\cong & & \downarrow & & \downarrow\cong \\ H_{\xi\eta} A^1_{\xi\eta} & \xleftarrow[\epsilon_\lambda]{\cong} & H_{\lambda\xi\eta} A^1_{\lambda\xi\eta} & \xrightarrow[\epsilon_{\xi\eta}]{} & H_\lambda A_\lambda{}^1 \end{array}.$$

The isomorphisms marked in the diagram come from (1), (2), (3), and (4), and we deduce from the diagram the isomorphism

$$(7) \qquad H_{\lambda\xi\eta} A^1_{\lambda\xi\eta} \xrightarrow[\epsilon_{\xi\eta}]{\cong} H_\lambda A_\lambda{}^1.$$

We remark that we could have proved (7) directly, by showing $\mathfrak{F}^1$ to be $(\xi\eta, \lambda)$-acyclic. But this would have involved a careful geometrical argument to ensure the contractibility of $|\operatorname{st} \sigma_\lambda| \cap X^1$ for these $\sigma_\lambda \subset \overline{X^1} - X^1$, which is the point where we must feed in the essential fact about second deriveds. In the above treatment, we fed in this fact comparatively painlessly in the retraction (3).

Combining (6) and (7), and observing that $A_\lambda{}^1 = D_\lambda{}^1$, the $\xi\eta$-augmentation induces an isomorphism

$$(8) \qquad H_\eta H_{\lambda\xi} D^1_{\lambda\xi\eta} \xrightarrow[\epsilon_{\xi\eta}]{\cong} H_\lambda D_\lambda{}^1.$$

Up to this point τ_μ has been fixed in L_μ. Now tensor the equation (8) by τ_μ and sum over all $\tau_\mu \in L_\mu$:

$$(9) \qquad H_\eta H_{\lambda\xi} D_{\lambda\mu\xi\eta} \xrightarrow[\epsilon_{\xi\eta}]{\cong} H_\lambda D_{\lambda\mu}.$$

Proceeding as in Lemma 6, the left-hand side of (9) is the E^1 term of the spectral sequence $E_{\mu,\eta}(H_{\lambda\xi} D_{\lambda\xi\eta})$, which therefore collapses, because the E^1

term is concentrated on the axis $q_\eta = 0$. Hence

$$H_{\mu\eta} H_{\lambda\xi} D_{\lambda\mu\xi\eta} \xrightarrow{\cong} H_\mu H_\eta H_{\lambda\xi} D_{\lambda\mu\xi\eta} \xrightarrow[\epsilon_{\xi\eta}]{\cong} H_\mu H_\lambda D_{\lambda\mu}, \tag{10}$$

the second isomorphism again by (9). The composition (10) is the isomorphism between E^2 terms that is sufficient to prove Lemma 18. The proof of Theorem 7 is complete.

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