ENGULFING

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The idea of an engulfing theorem is to convert a homotopy statement into a geometrical statement. For example let X^x be a compact PL (piecewise linear) subspace of an unbounded PL manifold M^m. The first engulfing theorem [4], [5], [8], [9] connected the following two properties:

(1) X is inessential in M (homotopy).

(2) X is contained in a ball in M (geometry).

The theorem said that the two properties were equivalent provided M is k-connected and

$$x \leqq m - 3,$$
$$2x \leqq m + k - 2.$$

Of course one way is trivial: if X is contained in a ball, then it is inessential in that ball. The other way is nontrivial, and was a key step in Stallings' proof of the Poincaré Conjecture [5], [8], and the proof of embedding theorems [3], [4].

When this result came to be generalised two different points of view emerged in response to different classes of problems. Stallings developed one point of view as a PL tool to solve topological problems, for example, the uniqueness of PL-structure of E^n, $n \geqq 5$ [6], the topological unknotting of spheres [7], and the approximation of homeomorphisms [1]. He envisaged the engulfing process as being like an amoeba U moving in the manifold so as to swallow up X. In particular if U is a small open cell this reduces to the first engulfing theorem above; more generally U can be an arbitrary open set provided that the pair (M, U) is sufficiently highly connected [6].

Meanwhile Zeeman [9], [11] developed the other point of view as a PL tool to be used within the PL category, in particular for embedding and isotopy theorems [3], [10], [11]. This approach envisaged engulfing in terms of regular neighbourhoods: given PL subspaces X, C we say that X *can be engulfed from* C if X lies in a regular neighbourhood of C. In particular if C is a point this means X is contained in a ball, and so we recover the first engulfing theorem above; more generally C can be an arbitrary PL subspace, possibly noncompact, satisfying a certain collapsibility condition (defined below), and provided (M, C) is sufficiently highly connected.

Now a regular neighbourhood D of C is characterised by three properties: D is a manifold, a neighbourhood of C, and $D \searrow C$ (D collapses to C). It transpires that only the third property matters from the point of view of engulfing, and more important than the first two properties is to make $D - C$ of minimal dimension. In fact we have an equivalent definition: *X can be engulfed from C* if there exists a PL subspace D such that

$$X \subset D \searrow C,$$
$$\dim (D - C) \leqq x + 1.$$

Intuitively one thinks of $D - C$ as a feeler pushed out from C so as to engulf X. The advantage of having the feeler of dimension only one more than that of X is apparent when engulfing singularities of maps is considered. For example, the feeler itself may introduce new singularities, but these will be of lower dimension than what we started with, and so amenable to attack by induction. The need for successive engulfings (by induction) raises problems in Stallings' approach, because his amoeba must move while it swallows; and there is a danger that during the second mouthful it may disgorge the first.

On the other hand the feeler approach enables successive feelers to be added without disturbing what has already been engulfed.

We state below an engulfing theorem that covers both approaches. We state it in feeler form because this allows us to cope with boundary problems, which require special treatment. Engulfing theorems which involve the boundary have proved useful for embedding [3] and compression [2] theorems. Irwin first proved a boundary engulfing theorem, and his result is a corollary of the theorem below.

DEFINITIONS. Let M be a PL manifold with boundary ∂M. Let C be a closed subspace. We say C is *q-collapsible* in M if there is a PL subspace Q such that $C \searrow Q$, $C - Q \subset \text{int } M$, and $\dim (Q \cap \text{int } M) \leqq q$. For example C is q-collapsible if $\dim C \leqq q$ or $C \subset \partial M$. Define X to be *C-inessential* if the inclusion map $X \subset M$ is homotopic in M, keeping $X \cap C$ fixed, to a map $X \rightarrow C$.

THEOREM. *Let M^m be a PL-manifold, with or without boundary, compact or not. Let C be a q-collapsible closed PL subspace, $q \leqq m-3$, and such that $\pi_i(M, C) = 0$, $i \leqq k$. Let X^x be a C-inessential compact PL subspace satisfying* (1) *or* (2):

(1) $\dim(X \cap \partial M) < x$, $x \leqq m-3$ *and* $m+k-2 \geqq \max\{2x, q+x\}$.

(2) $X \subset \partial M$ *and* $x \leqq m-4$, $2x \leqq m+k-3$, $q+x \leqq m+k-2$.

Then we can engulf X from C; that is to say there exists $D \subset M$ such that

$$X \subset D \searrow C,$$
$$\dim(D - C) \leqq x + 1,$$
$$D \cap \partial M = (X \cup C) \cap \partial M.$$

We recover the initial engulfing theorem by putting $\partial M = \varnothing$, C = point, $q = 0$ in case (1), and taking a regular neighbourhood of D, which gives a ball containing X. The connection with Stallings' theorem is given by the following lemma.

LEMMA. *Let M^m be a PL manifold without boundary, and X a compact subspace. Let C be a closed PL subspace, and U any open set containing C. If X can be engulfed from C (as in the thesis of the theorem) then there exists a PL homeomorphism $h: M \to M$, isotopic to the identity keeping C fixed, and having compact support, such that $hU \supset X$.*

The detailed proofs of the Theorem and the Lemma can be found in [11, Chapter 7] together with examples to show that the dimensional hypotheses in the theorem are best possible.

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