CORRECTION TO 'ON REGULAR NEIGHBOURHOODS'

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We are grateful to Ralph Tindall (**4**) for pointing out to us that our main uniqueness theorems in 'On relative regular neighbourhoods' ((**1**) Theorems 2 and 3) are false. The existence theorem ((**1**) Theorem 1) is true, and so are the uniqueness theorems for absolute regular neighbourhoods.

Tindall has an elegant counter-example. Let D^2 be a flat disc in $\mathbf{E}^4$. He constructs two regular neighbourhoods B_1 and B_2 of D^2 mod ∂D^2 in $\mathbf{E}^4$ such that B_1, D^2 is an unknotted ball-pair but B_2, D^2 is locally knotted at a point of ∂D^2. This means that there cannot be a homeomorphism of B_1 onto B_2 which is the identity on D^2.

He found the fallacy in our proof in (**1**) Lemma 6. In general $N_1 \neq U_{2r}$. What is needed is to add another condition to the definition of regular neighbourhood. The extra condition that we use was suggested by Homma, and some cases of the uniqueness theorem have been proved by Husch (**3**) using just such a condition.

In this paper we give a revised definition of relative regular neighbourhoods and give the complete proofs of existence and uniqueness theorems analogous to the theorems of (**1**). It turns out that the extra condition for regular neighbourhoods is trivially satisfied for regular neighbourhoods of polyhedra of codimension at least 3, and that in this case the theorems of (**1**) are true as they stand.

To avoid confusion the definitions and proofs will be given complete in this paper, avoiding all reference to (**1**).

Definitions and statements of results

Collapsing. A simplicial complex K *collapses simplicially* to a subcomplex K_0 if there exists a sequence of subcomplexes

$$K_0 \subset K_1 \ldots \subset K_r = K$$

such that, for each i, $K_i - K_{i-1}$ consists of a principal simplex of K_i together with a free face (i.e. a principal face which is not a face of any other simplex of K_i).

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A complex K *collapses* to K_0, written $K \searrow K_0$, if there exist subdivisions K', K_0', of K, K_0 such that K' collapses simplicially to K_0'. K is *collapsible* if it collapses to a single point.

A polyhedron X *collapses* to a subpolyhedron Y, written $X \searrow Y$, if there are triangulations K, K_0 of X, Y such that $K \searrow K_0$. X is *collapsible* if X collapses to a point.

Given subcomplexes K, L, M of some larger complex, M *link-collapses to* K *on* L if, for every simplex A of $\overline{M-L} \cap L$, $\operatorname{link}(A, \overline{M-L}) \searrow \operatorname{link}(A, \overline{K-L})$. K is *link-collapsible* on L if, for every simplex A in $\overline{K-L} \cap L$, $\operatorname{link}(A, \overline{K-L})$ is collapsible.

Given polyhedra X, Y, N in some PL space, N *link-collapses to* X *on* Y, if there are triangulations K, L, M of X, Y, N such that M link-collapses to K on L. X is *link-collapsible on* L if there are triangulations K, L, of X, Y such that K is link-collapsible on L.

Note. It follows from pseudo-radial projection (see below) that link-collapsing does not, in fact, depend on the particular choice of triangulation.

EXAMPLES

(1) A simplex is link-collapsible on any subcomplex.

(2) A manifold is link-collapsible on any subcomplex of the boundary.

(3) A cone is line-collapsible on any subcomplex of the base.

(4) If M is a locally flat submanifold of the manifold Q, with $\partial M \subset \partial Q$ and $\operatorname{Int} M \subset \operatorname{Int} Q$, then Q link-collapses to M on any subcomplex of ∂M.

Definition of regular neighbourhood. Let X, Y, N be subpolyhedra of a PL m-manifold M. N is *a regular neighbourhood of* X mod Y in M if it satisfies the conditions:

N1. N is an m-manifold;

N2. N is a topological neighbourhood of $X - Y$ in M and

$$N \cap Y = \operatorname{Frontier}(N) \cap Y = \overline{X-Y} \cap Y;$$

N3. $N \searrow \overline{X-Y}$;

N4. N link-collapses to $\overline{X-Y}$ on Y.

We say that N *meets the boundary regularly* if it also satisfies the condition:

N5. $\overline{(N \cap \partial M) - Y}$ is a regular neighbourhood of $X \cap \partial M$ mod $Y \cap \partial M$ in ∂M.

If N_1 is another regular neighbourhood of X mod Y in M, we say that N_1 is *smaller* than N if N is a topological neighbourhood of $N_1 - Y$ in M.

REMARK 1. The reason why $\overline{(N \cap \partial M) - Y}$ occurs in condition N5 is that in general $\overline{X \cap \partial M - Y \cap \partial M}$ is not the same as $\overline{X-Y} \cap \partial M$.

REMARK 2. Let N be a regular neighbourhood of X mod Y in M, and triangulate M with N, X, and Y as subcomplexes. If A is a simplex of $\overline{X-Y} \cap Y$, it follows immediately from the definition that link(A, N) is a regular neighbourhood of link(A, X) mod link(A, Y) in link(A, M).

Second derived neighbourhoods. If X, Y are polyhedra in M, a second derived neighbourhood of X mod Y in M is constructed as follows.

Choose a triangulation J of M which contains subcomplexes triangulating X and Y. Choose a second derived subdivision J'' of J (not necessarily barycentric second derived). Let $N = N(X-Y, J'')$, the closed simplicial neighbourhood of $X-Y$ in J'' (i.e. the union of the closed simplexes of J'' which meet $X-Y$).

Isotopy. An *isotopy* of N in M is a level-preserving PL embedding $F: N \times I \to M \times I$ (where I denotes the unit interval). So, for each t in I, there is an embedding $F_t: N \to M$ defined by $F(x,t) = (F_t x, t)$ for all x in N. If $N \subseteq M$, $F_0: N \to M$ is the inclusion map and $F_1 N = N_1$, we say that F is an isotopy in M *moving* N onto N_1. If $P \subseteq N$ and $F | P \times I$ is the identity we say that F *keeps* P *fixed.*

An *ambient isotopy* of M is a level-preserving PL homeomorphism $h: M \times I \to M \times I$ with $h_0: M \to M$ equal to the identity. If $N \subseteq M$ and $h_1 N = N_1$ we say that h *moves* N onto N_1. If $P \subseteq M$ and $h | P \times I$ is the identity we say that h *keeps* P *fixed.*

We can now state the main theorems.

Let X, Y be polyhedra in the PL m-manifold M.

THEOREM 1 (*Existence*). *If X is link-collapsible on Y, then any second derived neighbourhood N of X* mod *Y is a regular neighbourhood of X* mod *Y in M. If, further, $X \cap \partial M$ is link-collapsible on $Y \cap \partial M$, then N meets the boundary regularly.*

THEOREM 2 (*Uniqueness*). *Let N_1, N_2 be regular neighbourhoods of X* mod *Y in M. Then there exists a small regular neighbourhood N_3 of X* mod *Y in M and a PL homeomorphism of N_1 onto N_2 keeping N_3 fixed. In fact there is an isotopy in M throwing N_1 onto N_2 and keeping N_3 fixed.*

THEOREM 3 (*Uniqueness*). *Let N_1, N_2, N_3 be regular neighbourhoods of X* mod *Y in M, meeting the boundary regularly and N_3 being smaller than both N_1 and N_2. Let P be the closure of the complement of a second derived neighbourhood of $N_1 \cup N_2$* mod *Y in M. Then there is an ambient isotopy of M throwing N_1 onto N_2 and keeping $N_3 \cup P$ fixed.*

Furthermore, if $N_1 \cap \partial M = N_2 \cap \partial M$ we may insist that the ambient isotopy keeps ∂M fixed.

REMARKS

(i) The link-collapsibility is not a necessary condition for the derived neighbourhood to be a regular neighbourhood.

Example. Let K be a 'dunce hat' (see (**6**) Chapter 3). Then K is not collapsible but any regular neighbourhood of K in a 3-manifold is collapsible. Now let X be the cone on the dunce hat, embedded in $\mathbf{E}^4$, and let Y be the vertex of the cone. Then any second derived neighbourhood of X mod Y in $\mathbf{E}^4$ will be a regular neighbourhood of X mod Y although X is not link-collapsible on Y.

(ii) The theory could be generalized even further by replacing condition N1 for a regular neighbourhood by the condition

N1a. $N - Y$ is an m-manifold.

The existence theorem would then hold without the link-collapsible condition and the uniqueness theorems would then hold as they stand. The proofs would be identical.

The fourth condition looks rather unpleasant to verify for applications but it is frequently superfluous in view of the following lemma proved in (**2**).

LEMMA. *Let N be a* PL *n-manifold, X a polyhedron in N and Y a subpolyhedron of $X \cap \partial N$. If X is link-collapsible on Y and* $\dim X \leqslant n-3$, *then N link-collapses to X on Y.*

Proofs of the main theorems

LEMMA 1. *If $X \subset M$ and X is collapsible, then any regular enlargement of X in M is a ball.*

Proof. Let N be the regular enlargement. By (**5**) Theorems 4 and 7 we can choose triangulations J, K, L of M, X, N such that L collapses simplicially to K and K collapses simplicially to a point. Then, by (**5**) Theorem 23, Corollary 1, L is a combinatorial ball.

Full subcomplexes. If L is a subcomplex of K, L is *full* in K if no simplex of $K - L$ has all its vertices in L.

REMARKS

(i) If L is full in K, any simplex of K meets L in a face or the empty set.

(ii) If L is *any* subcomplex of K and K', L' are first derived subdivisions, then L' is full in K'.

(iii) If L is full in K and K', L' is *any* subdivision of K, L then L' is full in K'.

Well situated. We introduce a technical term for convenience in the proof of Theorem 1. Let J be a combinatorial manifold and K and L

finite subcomplexes. We say that K and L are *well situated* in J if:

(1) $K \cup L$ and $\overline{K-L}$ are full in J;

(2) for every simplex A in $N(K-L, J)-K$, $\text{link}(A, J)$ meets $\overline{K-L}$ in a single simplex; and

(3) $\overline{K-L} \cup \partial J$ is full in J.

LEMMA 2. *Suppose $K, L \subset J$, and $N = N(K-L, J)$. If K, L are well situated in J then:*

(i) $N \searrow (N \cap \partial J) \cup \overline{K-L} \searrow \overline{K-L}$; *and*

(ii) *for every simplex A in N not meeting $\overline{K-L}$,*

$$\text{link}(A, N) \searrow \text{link}(A, N) \cap (\overline{K-L} \cup \partial J) \searrow \text{link}(A, N) \cap \overline{K-L}.$$

Proof.

(i) This is a modification of (**1**) Lemma 2. Let A_1, A_2, ..., A_n be the simplexes of N which do not meet $\overline{K-L}$ and suppose that A_1, A_2, ..., A_r lie in $\text{Int}\, J$ and are in order of decreasing dimension and A_{r+1}, ..., A_n lie in ∂J and are in order of decreasing dimension. For each i, link (A_i, J) meets $\overline{K-L}$ in a single simplex, B_i say.

Now $A_i B_i$ collapses simplicially onto $\dot{A}_i B_i$. Since each A_i precedes its faces, we get $N \searrow F \searrow \overline{K-L}$, where

$$F = \overline{K-L} \cup \bigcup_{r+1}^{n} A_i B_i.$$

It remains to show that $F = (N \cap \partial J) \cup \overline{K-L}$.

If $C \in N \cap \partial J$, then C is contained in some $A_i B_i$ for $i > r$, and so $C \in F$. So $(N \cap \partial J) \cup \overline{K-L} \subseteq F$. Conversely, if $i > r$, then $A_i B_i$ has all its vertices in $\overline{K-L} \cup \partial J$, which is full in J, and so $A_i B_i \in \overline{K-L} \cup \partial J$. But $A_i \notin \overline{K-L}$, and so $A_i B_i \in \partial J$. So $F = (N \cap \partial J) \cup \overline{K-L}$.

(ii) If $A \in N$ and does not meet $K-L$, let $J^* = \text{link}(A, J)$, $K^* = K \cap J^*$, $L^* = L \cap J^*$, $N^* = \text{link}(A, N)$. Then K^*, L^* are well situated in J^* and $N^* = N(K^*-L^*, J^*)$. The result now follows directly from the proof of part (i).

LEMMA 3. *Suppose that K, L are well situated in the combinatorial manifold J, and that K is link-collapsible in L. Then:*

(i) *$N = N(K-L, J)$ is a regular neighbourhood of K mod L in J; and*

(ii) *N is a regular neighbourhood of $K \cup (N \cap \partial J)$ mod $L \cup \overline{\partial J - N}$ in J.*

Proof. Conditions N2 and N4 for a regular neighbourhood follow directly from Lemma 2. Condition N3 follows from the definition of a closed simplicial neighbourhood. It remains only to show that N is a combinatorial manifold. This is proved by induction on the dimension of J.

Let x be a vertex of N.

Case a, $x \in K - L$. Then $\text{link}(x, N) = \text{link}(x, J)$ is a sphere or ball.

Case b, $x \in \overline{K-L} \cap L$. In the notation of Lemma 2 part (ii), link $(x, N) = N(K^* - L^*, J^*)$, and K^*, L^* are well-situated in J^*. Now $K \cup L$ and $\overline{K-L}$ are full in J and so $K^* \cup L^* = \text{link}(x, K \cup L)$ and $\overline{K^* - L^*} = \text{link}(x, \overline{K-L})$. So K^* is link-collapsible on L^*. By the inductive hypothesis, link (x, N) is a regular neighbourhood of K^* mod L^* in J^* and therefore a regular enlargement of $\overline{K^* - L^*}$. By the link-collapsibility, $\overline{K^* - L^*} = \text{link}(x, \overline{K-L})$ is collapsible and so, by Lemma 1, $\text{link}(x, N)$ is a combinatorial ball.

Case c, $x \in N - K$. In the same notation as above, K^* is in this case a single simplex. A simplex is link-collapsible on any subcomplex, and so we can apply the inductive hypothesis together with Lemma 1, as in case b, to show that $\text{link}(x, N)$ is a combinatorial ball.

Proof of Theorem 1. Let K, $L \subset J$ be the triangulations of X, $Y \subset M$, and let K'', $L'' < J''$ be the second derived subdivisions such that $N = |N(K'' - L'', J'')|$.

K'', L'', and J'' are obtained from first derived subdivisions K', L', and J', by starring all the simplexes of J' in some order of decreasing dimension. Let J^* be obtained from J' by starring the simplexes of $J' - (\overline{K' - L'})$ in order of decreasing dimension at the same points as were used for J''. Then we have

LEMMA 4

(i) $N = |N(X - Y, J^*)|$;

(ii) *K^*, L^* are well situated in J^*.*

Proof. J^* is obtained from J by stellar-subdividing the simplexes of $\overline{K^* - L^*}$ in order of decreasing dimension. Let B denote a typical simplex of $K^* - L^*$ and let $\hat{B}$ denote its point of subdivision. Then

$$|N(X - Y, J^*)| = \bigcup_B |N(B - \dot{B}, J'')| = \bigcup_B |N(\hat{B}, J'')| = N.$$

Conditions (1) and (3) are true in J' and remain true in J^*. Condition (2) follows from (5) Lemma 4.

The first half of Theorem 1 now follows from Lemma 3. Applying this result in the boundary gives the second part of the theorem.

Derived neighbourhoods. Let K and L be subcomplexes of the combinatorial manifold J, such that $K \cup L$ and $\overline{K-L}$ are full in J. Let K', L', and J' be first derived triangulations. Then $N = N(K' - L', J')$ is a *derived neighbourhood* of K mod L in J.

If X, Y are subpolyhedra of a PL manifold M, a derived neighbourhood of X mod Y in M is defined by first choosing triangulations K, L, and J of X, Y, and M such that $K \cup L$ and $\overline{K-L}$ are full in J and then taking a derived neighbourhood of K mod L in J. In particular any second derived neighbourhood of X mod Y in M is a derived neighbourhood. If L is empty, we talk of a derived neighbourhood of X in M.

Uniqueness of derived neighbourhoods

LEMMA 5 ((**6**) Chapter 3, Lemmas 14, 15). *If N_1, N_2 are any two derived neighbourhoods of X in M, then there is a PL ambient isotopy of M, fixed on X, throwing N_1 onto N_2.*

Shelling. Suppose that N_1 and N are PL m-manifolds (with boundary), $N_1 \subset N$. *N shells to N_1* if there is a finite sequence $N_1 \subset N_2 \subset \ldots \subset N_k = N$ of submanifolds of N such that, for each i, $\overline{N_i - N_{i-1}} = B_i$, say, is a PL m-ball and $B_i \cap \partial N_i = \partial B_i \cap \partial N_i$ is a PL $(m-1)$-ball.

LEMMA 6. *Let N be a regular neighbourhood of X in M. Let N_1 be a derived neighbourhood of X in N. Then N shells to N_1.*

Proof. By condition N1 for a regular neighbourhood, there are triangulations K, J of X, N, such that J collapses simplicially to K. Let K'', J'' be the barycentric second derived subdivision. Then, by a result of Whitehead ((**5**) Lemma 11) J'' shells to $N(K'', J'')$. But, by Lemma 5, there is a PL homeomorphism $h\colon N \to N$, throwing $N(K'', J'')$ onto N_1. Therefore N shells to N_1.

Pseudo-radial projection. In order to produce a relativization of Lemma 6 it is necessary to look closely at the links of some of the simplexes. We shall require the technique of 'pseudo-radial' projection.

Suppose that A is a simplex in the simplicial complex K. Let K' be some division of K and let B be a simplex of K' whose interior lies in the interior of A. Let $L = \text{link}(B, A')$, A' being the subdivision of A induced by the subdivision K'.

Now the join $L.\text{link}(A, K)$ may be regarded as being linearly embedded in $A.\text{link}(A, K)$ in the obvious way. We wish to define a PL homeomorphism $H\colon \text{link}(B, K') \to L.\text{link}(A, K)$. Projecting radially from B gives a homeomorphism which is not PL. We approximate it by a PL map as follows. First choose a subdivision β of $\text{link}(B, K')$ such that, for each simplex C in $L.\text{link}(A, K)$, the intersection $B.C \cap \text{link}(B, K')$ is triangulated as a subcomplex of $\beta.\text{link}(b, K')$. Now project each vertex of β $\text{link}(B, K')$ radially from B to $\text{L.link}(A, K)$ and join up linearly. This gives the required PL homeomorphism.

Notice that, if K_1 is any subcomplex of K containing A, then h sends $\text{link}(B, K_1') \to L.\text{link}(A, K_1)$.

This immediately implies that the conditions of link-collapsing defined at the beginning of the paper are independent of the triangulation.

LEMMA 7. *Let N be a regular neighbourhood of X mod Y in M. Let N_1 be a derived neighbourhood of $\overline{X-Y}$ mod $\overline{X-Y} \cap Y$ in N. Then N shells to N_1.*

Proof. Let K, L, J be the simplicial complexes triangulating $\overline{X-Y}$, $\overline{X-Y} \cap Y$, and N, and let K', L', and J' be the first derived subdivisions, such that $N_1 = N(K'-L', J')$. Let $A_1, A_2, \ldots, A_r$ be the simplexes of L in order of increasing dimension. Let J^* be the subdivision of J obtained by starring the simplexes of $J-K$ in order of decreasing dimension at the same subdivision points as for J'. Then $N_1 = N(K-L, J^*)$. Let $L_i = \bigcup_{j=1}^{i} A_j$ and let $U_i = N(K-L_i, J^*)$. Now $U_0 = N(K, J^*)$, which is a derived neighbourhood of K in J, and so N shells to U_0 by Lemma 6. $U_r = N_1$, and so we only need to prove that U_{i-1} shells to U_i for each i. Now $U_{i-1} - U_i$ is the set of simplexes of J^* which meet $\text{Int}\, A_i$ but do not meet $K - L_i$. Let $J^\wedge = \text{link}(A_i, J^*)$, $K^\wedge = \text{link}(A_i, K)$. Then

$$\overline{U_{i-1} - U_i} = A_i.J^\wedge \cap U_{i-1} \quad \text{and} \quad A_i.J^\wedge \cap U_i = A_i.N(K^\wedge, J^\wedge).$$

So we must show that $A_i.J^\wedge$ shells to $A_i.N(K^\wedge, J^\wedge)$. For this it is sufficient to show that $J^\wedge$ shells to $N(K^\wedge, J^\wedge)$. Now consider the pseudo-radial projection $J^\wedge \to \text{link}(A, J)$. This throws $N(K^\wedge, J^\wedge)$ onto a first derived neighbourhood of $\text{link}(A_i, K)$ in $\text{link}(A_i, J)$. J is a regular neighbourhood of K mod L, and so $\text{link}(A_i, J)$ is a regular neighbourhood of link (A_i, K). So, by Lemma 6, $\text{link}(A_i, J)$ shells to any derived neigbourhood of link (A_i, K) in $\text{link}(A_i, J)$, and, by the pseudo-radial projection, $J^\wedge$ shells to $N(K^\wedge, J^\wedge)$. This completes the proof of Lemma 7.

Proof of Theorem 2. We are given two regular neighbourhoods N_1, N_2 of X mod Y in M. Triangulate M so that N_1, N_2, X, and Y are subcomplexes, $\overline{X-Y}$, $X \cup Y$ being full, and let N be a first derived neighbourhood of X mod Y with respect to this triangulation. Let N_3 be a smaller derived neighbourhood of X mod Y in M (i.e. derived neighbourhood such that $N_3 - Y$ is contained in the interior of N as a subset of M).

Now we know from Lemma 7 that N_1 and N_2 both shell to N. We shall produce isotopies of N in M, keeping N_3 fixed, and throwing N_3 onto either N_1 or N_2. Composing these will give the required isotopy in M, throwing N_1 onto N_2. Now N_1 shells to N. So there are submanifolds

$N = U_0 \subset U_1 \subset \ldots U_r = N_1$ such that, for each i, $\overline{U_i - U_{i-1}} = B_i$ is a PL m-ball and $B_i \cap \partial U_i$ a PL $(m-1)$-ball in ∂B_i. Let $F_i = B_i \cap U_{i-1} = \partial B_i \cap \partial U_{i-1}$. Then F_i is also an $(m-1)$-ball and, since N_3 is smaller than N, $F_i \cap N_3 \subseteq F_i \cap L \subseteq \partial F_i$. Let C_i be a derived neighbourhood of F_i mod ∂F_i in U_{i-1}. Then C_i is an m-ball, by Lemma 1, meeting B_i in the common face F_i, and $\operatorname{Int} C_i$ does not meet N_3. Then $B_i \cup C_i$ is also a PL m-ball, and there is a PL homeomorphism $h_i \colon B_i \cup C_i \to C_i$ which is fixed on their common face $\overline{\partial C_i - F_i}$. Moreover, this homeomorphism can be realized by a PL isotopy in $B_i \cup C_i$ keeping the face $\overline{\partial C_i - F_i}$ fixed throughout. Thus we have an isotopy in M, fixed on N_3, throwing U_i onto U_{i-1}. Composing these gives the required isotopy in M throwing N_1 onto N. We can do the same construction for N_2.

ADDENDUM TO THEOREM 2. *If N is a regular neighbourhood of X* mod *Y in M and if N' is a derived neighbourhood of X* mod *Y in N, then N' is a regular neighbourhood of X* mod *Y in M.*

[*N.B.* There is no assumption that X should be link-collapsible on Y.]

Proof. As in the proof of Theorem 2, there is a PL homeomorphism which is the identity on $\overline{X - Y}$ and throws N onto N'.

Proof of Theorem 3. We give the proof as a series of lemmas. First a special case of Theorem 3.

LEMMA 8. *Let N_1 and N_2 be regular neighbourhoods of X* mod *Y in M and suppose that $N_1 \cap \partial M = N_2 \cap \partial M = \overline{X - Y} \cap \partial M$. Then there is a PL ambient isotopy of M, fixed on $X \cup Y \cup \partial M$, throwing N_1 onto N_2.*

Proof. As in the proof of Theorem 2, triangulate M with X, Y, N_1, and N_2 as subcomplexes and let N be a second derived neighbourhood of X mod Y in M. Then we know that N_1 and N_2 both shell on N.

Now N_1 shells to N and so we have submanifolds

$$N = U_0 \subset U_1 \ldots \subset U_r = N_1,$$

with $\overline{U_i - U_{i-1}} = B_i$ and $B_i \cap U_{i-1} = F_i$. Now

$$N \cap \partial M \subseteq N_1 \cap \partial M \subseteq \overline{X - Y} \cap \partial M \subseteq N \cap \partial M.$$

So $N \cap \partial M = N_1 \cap \partial M$ and so $B_i \cap (X \cup Y \cup \partial M) \subseteq \partial F_i$. Now let C_i be a second derived neighbourhood of F_i mod ∂F_i in U_{i-1} and let D_i be a second derived neighbourhood of $\overline{\partial B_i - F_i}$ mod ∂F_i in $\overline{M - U_i}$. Choosing these derived neighbourhoods with respect to triangulations having X and Y as subcomplexes ensures that they will not meet $X \cup Y \cup \partial M$ except possibly in points of ∂F_i. Now C_i and D_i are m-balls, by Theorem 1 and Lemma 1; so $C_i \subseteq B_i \cup C_i \subseteq B_i \cup C_i \cup D_i$ are all PL m-balls with the face $\overline{\partial C_i - F_i}$ in common. So there is a PL ambient isotopy of M, fixed outside $B_i \cup C_i \cup D_i$, which throws $B_i \cup C_i$ onto C_i. Thus the ambient isotopy is fixed on $X \cup Y \cup \partial M$ and throws U_i onto U_{i-1}. Composing gives an ambient isotopy throwing N_i onto N, and we can do the same construction for N_2.

LEMMA 9. *Let N_1 and N_2 be regular neighbourhoods of X* mod *Y in M meeting the boundary regularly. Then there is an ambient isotopy of M, fixed on $X \cup Y$ throwing $N_1 \cap \partial M$ onto $N_2 \cap \partial M$.*

Proof. Put $N_1' = \overline{N_1 \cap \partial M - Y \cap \partial M}$ and $N_2' = \overline{N_2 \cap \partial M - Y \cap \partial M}$. Now apply Lemma 8 to N_1' and N_2' in ∂M. This yields an ambient isotopy of ∂M, fixed on $(X \cup Y) \cap \partial M$, and throwing N_1' onto N_2'. We must extend this ambient isotopy to the rest of M. In fact the proof of Lemma 8 gives the ambient isotopy as a composition of isotopies each supported by a ball. If $P_1, P_2, \ldots, P_k$ are the balls supporting these isotopies of ∂M, the construction of Lemma 8 ensures that, for each i, $P_i \cap (X \cup Y) \subseteq \partial P_i$. Now let Q_i be a second derived neighbourhood of P_i mod ∂P_i in M with respect to a triangulation having X and Y as subcomplexes. Then $\operatorname{Int} Q_i$ does not meet $X \cup Y$. An ambient isotopy of ∂M fixed outside P_i may now be extended to an ambient isotopy of M fixed outside Q_i. Composing these isotopies gives the required ambient isotopy of M, fixed on $X \cup Y$, and throwing $N_1 \cap \partial M$ onto $N_2 \cap \partial M$.

LEMMA 10. *If N is a regular neighbourhood of X* mod *Y in N meeting the boundary regularly, then N is a regular neighbourhood of*

$$X \cup (N \cap \partial M) \text{ mod } Y \cup (N \cap \partial M),$$

where N denotes the frontier of N in M.

Proof. Let $N' = \overline{N \cap \partial M - Y}$. N' is a regular neighbourhood of $X \cap \partial M$ mod $Y \cap \partial M$ in ∂M. It follows immediately from the definitions that N' is a regular neighbourhood of $X \cap \partial M$ mod $Y \cap \partial M$ in ∂N. Now suppose that N_1 is any other regular neighbourhood of X mod Y in M meeting ∂M regularly, and let $N_1' = \overline{N_1' \cap \partial M - Y}$. By Theorem 2, there is a PL homeomorphism $h\colon N_1 \to N$, which is the identity on $\overline{X - Y}$. hN_1' will then be a regular neighbourhood of $X \cap \partial M$ mod $Y \cap \partial M$ in ∂N. By Lemma 9, we may replace h by another homeomorphism $h'\colon N_1 \to N$, such that h' is also the identity on $\overline{X - Y}$, and $h'N_1' = N'$.

It follows that if the result of the present lemma holds for N_1 then it must also hold for N. We shall take the special case when N_1 is a second derived neighbourhood of X mod Y in M. N_1 is a regular neighbourhood of X mod Y in M meeting the boundary regularly by the addendum to Theorem 2. We must prove that N_1 is a regular neighbourhood of $X \cup N_1'$ mod $Y \cup \dot{N}_1'$. From the remarks above, N_1 is an m-manifold and N_1' is an $(m-1)$-manifold. A second derived neighbourhood must always satisfy the second condition for a regular neighbourhood. There remain only conditions N3 and N4. Now, as in the proof of Theorem 1, let K, L, and J be triangulations of X, Y, and M and let K'', L'', and J'' be their second derived subdivisions such that $N_1 = N(K'' - L'', J'')$.

Let J^* be the subdivision of the first derived J' obtained by starring only the simplexes of $J' - \overline{K' - L'}$ at the same subdivision points as for J''. Then, by Lemma 4, $N_1 = N(K^* - L^*, J^*)$ and K^* are well situated in J^*.

It follows from Lemma 2 that $N_1 \searrow N_1 \cap (\partial J^* \cup \overline{K^* - L^*}) = N_1' \cap \overline{X - Y}$.

From the second part of Lemma 2, N_1 link-collapses to $\overline{X - Y} \cup N_1'$ on $(\overline{X - Y} \cap Y) \cup \dot{N}_1'$. So conditions N3 and N4 for a regular neighbourhood are satisfied.

LEMMA 11. *Let N_1 and N_2 be regular neighbourhoods of X* mod *Y in M meeting the boundary regularly and such that $N_1 \cap \partial M = N_2 \cap \partial M$. Then there is a PL ambient isotopy of M, fixed on $X \cup Y \cup \partial M$, throwing N_1 onto N_2.*

Proof. By Lemma 10, both N_1 and N_2 are regular neighbourhoods of $X \cup (N_1 \cap \partial M)$ mod $Y \cap (\dot{N}_1 \cap \partial M)$ in M, and the result follows by applying Lemma 8.

LEMMA 12. *Let N_1 and N_2 and N_3 be regular neighbourhoods of X* mod *Y in M, each meeting the boundary regularly, and N_3 being smaller than both N_1 and N_2. Let P be the closure of the complement of a second derived neighbourhood of $N_1 \cup N_2$* mod *L in M. Then N_1 and N_2 are both regular neighbourhoods of N_3* mod *P, meeting the boundary regularly.*

Proof. Applying Lemmas 9 and 11 to N_1, we see that there is a PL homeomorphism $h\colon N_1 \to N_1$, throwing N_3 onto a second derived neighbourhood of $\overline{X - Y}$ mod $\overline{X - Y} \cap Y$ in N_1. So, by Lemma 6, N_1 shells to N_3. So N_1 collapses to N_3.

We must show that N_1 link-collapses to N_3 on P. This is the same as saying that N_1 link-collapses to N_3 on $N_3 \cap P$ which is equal to $\overline{X - Y} \cap Y$. Now triangulate M with X, Y, N_1, N_2, and N_3 as subcomplexes. Let A be a simplex of $\overline{X - Y} \cap Y$. Then $\mathrm{link}(A, N_1)$ and $\mathrm{link}(A, N_2)$ are regular neighbourhoods of $\mathrm{link}(A, X)$ mod $\mathrm{link}(A, Y)$ in $\mathrm{link}(A, M)$. (See Remark 2 after the definition of regular neighbourhoods.) So we can apply the argument above to deduce that N_1 link-collapses to N_3 on $\overline{X - Y} \cap Y$.

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