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What's wrong with Euclid Book V

(*The 2007 David Crighton Lecture*)

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Abstract

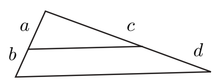
Not having the real numbers, Euclid defined ratios abstractly in Book V for use in geometric theorems, but failed to define ratios of ratios, in effect blocking certain further developments of Greek mathematics. We introduce a new axiom that can be used instead of subtraction, not only to prove the propositions of Book V but also to define ratios of ratios.

1. *Introduction*

The Theory of Proportion of Eudoxus has long been regarded as one of the finest, if not *the* finest, achievement of ancient Greek mathematics.

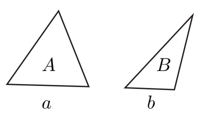
The Greeks were handicapped by not having the real numbers. Consequently they were unable, for example, to define the ratio of two lines as we do now, by choosing a unit, measuring the length of the lines relative to that unit as real numbers, and then dividing one real number by the other. A similar problem held for the ratio of two areas or two volumes. The Pythagoreans had used rational ratios for commensurable lengths, but were unable to handle the ratio of noncommensurables (see [1]). Eudoxus overcame the difficulty by defining ratios abstractly, and *Book V* of Euclid's *Elements* [3] is an exposition of his work. In fact, *Book V* is the first known example of abstract algebra. It enabled the Greeks to state and prove rigorously geometrical theorems involving ratios. For example, we have the following statements.

(i) *Euclid* (*Book VI*, Proposition 2). *A line parallel to the base of a triangle divides the sides in equal ratios.*



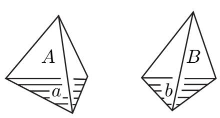
$$a : b = c : d$$

(ii) *Euclid* (*Book VI*, Proposition 1). *If two triangles have the same height then the ratio of their areas equals the ratio of their bases.*



$$A : B = a : b$$

(iii) *Euclid* (*Book XII*, Proposition 5). *If two tetrahedra have the same height then the ratio of their volumes equals the ratio of their base areas.*



$$A : B = a : b$$

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In his theory of proportion Eudoxus also introduced the concept of the quantifiers '$\exists$' (there exists) and '$\forall$' (for all). Over 2000 years later, at the beginning of the 19th century, *Book V* inspired Bolzano to introduce $\exists, \forall$ into analysis, and in turn his work inspired Dedekind to discover the real numbers on 24 November 1854. (The symbols '$\exists$' and '$\forall$' are themselves a modern notation of the 20th century. Jeremy Gray tells me that G. Peano introduced $\exists$ in 1897, and G. Gentzen introduced $\forall$ in 1935.)

My main criticism of Euclid's treatment, however, is that he forgot to define the ratio of two ratios. More precisely, he was unable to do so because of his tactical use of subtraction in the proof of Proposition 8, on which most of the other propositions of *Book V* depend. This had serious strategic consequences for the whole of Greek mathematics, including the following.

(i) It prevented them from being able to define cross-ratios:

$$(ABCD) = \frac{AB.CD}{AD.CB} = (AB : AD) : (CB : CD).$$

Hence they were unable to develop projective geometry.

(ii) It prevented them from being able to define products of ratios:

$$(a : b)(c : d) = (a : b) : (d : c).$$

Hence they were unable to develop group theory.

(iii) It prevented them from being able to define acceleration:

$$\text{velocity} = \frac{\text{distance}}{\text{time}}; \quad \text{acceleration} = \frac{\text{change in velocity}}{\text{time}}.$$

Hence they were unable to develop dynamics.

The problem that Euclid faced was the *additive structure*, and the resulting concept of *subtraction* $a - b$ (given $a > b$). He assumed that any set of magnitudes had an additive structure, which he then used to prove the desired properties of ratios. But he failed to develop an additive structure on the set of ratios themselves, and so he could not repeat the process to define ratios of ratios.

Perhaps Euclid may have been prejudiced in favour of the additive structure because of its usefulness in his own favourite construction, the Euclidean algorithm. Of course one needs the additive structure for integers, but it is not so obviously necessary for other magnitudes such as areas, volumes or musical notes.

In this paper we shall solve the problem by introducing a new axiom, that can be used instead of subtraction to prove Proposition 8. We focus on Proposition 8 because this is the unique proposition in *Book V*, with a statement that does not involve the additive structure, but with a proof that does. Also, most of *Book V* depends upon it (see Section 4, Figure 1). Hence the new axiom enables us to bypass the problem of subtraction.

One can also show that the ratios themselves satisfy the new axiom, and so the set of ratios becomes itself another set of magnitudes, from which we can then use the same construction to define ratios of ratios, as desired.

2. *Axiomatic approach*

In this section we develop the theory of proportion axiomatically, using modern notation. Let $\mathbb{N}$ denote the positive integers.

2.1. *Axioms for a set of magnitudes*

DEFINITION OF A SET OF MAGNITUDES. A set M of *magnitudes* is a set, ordered by size, with $\mathbb{N}$-action, satisfying the following axioms.

Order: Given $a, b \in M$, then either a is smaller than b, written $a < b$, or a is the same size as b, written $a \sim b$, or a is larger than b, written $a > b$. (We distinguish between $a \sim b$ and $a = b$ as follows. If a, b are intervals, for example, we write $a \sim b$ if they have the same length, and $a = b$ if they are the same interval. Similarly, if a, b are regions in the plane we write $a \sim b$ if they have the same area, and $a = b$ if they are the same region.)

Equivalence: $\sim$ is an equivalence relation:

$$\begin{array}{ll} \text{reflexive:} & a \sim a; \\ \text{symmetric:} & a \sim b \Longrightarrow b \sim a; \\ \text{transitive:} & a \sim b \sim c \Longrightarrow a \sim c. \end{array}$$

Opposites: $a < b \Longleftrightarrow b > a$.

Transitivity: $a < b < c \Longrightarrow a < c$;
$a < b \sim c \Longrightarrow a < c$;
$a \sim b < c \Longrightarrow a < c$.

$\mathbb{N}$-*action*: For all $a \in M, n \in \mathbb{N}$, there exists $na \in M$ such that

$$\begin{aligned} 1a &= a; \\ m(na) &= (mn)a; \\ a \sim b &\Longrightarrow na \sim nb. \end{aligned}$$

$\mathbb{N}$-*action and order*: $a < b \Longrightarrow na < nb$;
$m < n \Longrightarrow ma < na$.

Archimedean axiom[†]: For all $a, b \in M$, there exists $n \in \mathbb{N}$ such that $a < nb$.

New axiom: For all $a, b \in M$, $a < b \Longrightarrow$ there exists $n \in \mathbb{N}$ such that $(n+1)a < nb$.

This completes the definition and axioms for a set of magnitudes.

Examples of sets of magnitudes.

(i) the positive integers $\mathbb{N}$;
(ii) the positive rationals $\mathbb{Q}_+$;
(iii) the positive reals $\mathbb{R}_+$;
(iv) intervals;
(v) areas;
(vi) volumes;
(vii) musical notes.

Remark 1. Notice that we have not included any additive structure in the definition of a set of magnitudes because we shall develop the theory without it. Euclid assumed that any set of magnitudes had an additive structure, including subtraction, because he needed the latter for his proof of Proposition 8. In Section 5 we shall give both his proof of Proposition 8 and a new proof that uses the new axiom instead of the additive structure. Meanwhile, we shall give a definition of additive structure in the next section, Section 3.

Our next task is construct the set $X(M)$ of ratios of a set M of magnitudes. But first we state the axioms for a set of ratios.

[†]This axiom is traditionally called the Archimedean axiom although Euclid included it as Definition 4 in *Book V*, which he probably published before Archimedes was born. Their dates are: Euclid c. 330–270 BC and Archimedes 287–212 BC.

2.2. *Axioms for a set of ratios*

Definition of a set of ratios. A set X of *ratios* is a set, ordered by size, with a unit 1 and $\mathbb{Q}_+$-action, satisfying the following axioms.

Order: For all $x, y \in X$, either $x < y$, or $x = y$, or $x > y$. (There is no need to introduce $x \sim y$ separate from $x = y$.)

Opposites: $x < y \Longleftrightarrow y > x$.

Transitivity: $x < y < z \Longrightarrow x < z$.

$\mathbb{Q}_+$-*action*: For all $x \in X,\ r \in \mathbb{Q}_+$, there exists $rx \in X$ such that

$$1x = x;$$
$$r(sx) = (rs)x.$$

$\mathbb{Q}_+$-*action & order*: $x < y \Longrightarrow rx < ry$;
$r < s \Longrightarrow rx < sx$.

Archimedean axiom: For all $x, y \in X$, there exists $n \in \mathbb{N}$ such that $x < ny$.

New axiom: For all $x, y \in X$, $x < y \Longrightarrow$ there exists $n \in \mathbb{N}$ such that $(n+1)x < ny$.

This completes the definition and axioms for a set of ratios.

Remark 2. Notice that by forgetting the non-integer actions on a set of ratios, the latter automatically satisfies the axioms for a set of magnitudes. This is the crucial observation that will enable us to construct ratios of ratios.

We must now define the set $X(M)$ of ratios of a given set M of magnitudes. The key idea is Definition 5 of *Book V*. We first give its English translation, as translated from the Greek by T. L. Heath in 1908 [**3**, Vol. 2, p. 114].

Book V, Definition 5

> Magnitudes are said to *be in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

We now rewrite this in modern notation as follows.

Definition of proportional. Let M be a set of magnitudes, and let $a, b, c, d \in M$. Define (a, b) to be *proportional to* (c, d), written $(a, b) \sim (c, d)$, if

$$\text{for all } m, n \in \mathbb{N}, \quad ma <, \sim, > nb \quad \text{as } mc <, \sim, > nd.$$

It is easy to verify that proportionality is an equivalence relation:

$$\begin{aligned} &\text{reflexive:} && (a, b) \sim (a, b); \\ &\text{symmetric:} && (a, b) \sim (c, d) \Longrightarrow (c, d) \sim (a, b); \\ &\text{transitive:} && (a, b) \sim (c, d) \sim (e, f) \Longrightarrow (a, b) \sim (e, f). \end{aligned}$$

Definition of ratio. The *ratio* $a : b$ is defined to be the equivalence class of (a, b). Thus, if two pairs are proportional, this is equivalent to to saying that they are in the same ratio.

Definition of the set $X(M)$ of ratios of M. Define

$$X(M) = \{a : b\, ; a, b \in M\}.$$

Definition of order in $X(M)$ (*Book V*, Definition 7). Define

$$a : b < c : d$$

if there exist m, n such that

$$ma < nb \quad \text{and} \quad mc \gtrsim nd$$

or

$$ma \sim nb \quad \text{and} \quad mc > nd.$$

Define

$$a : b > c : d \quad \text{if } c : d < a : b.$$

Definition of Q_+-action. Given $x \in X(M)$ and $r \in \mathbb{Q}_+$, choose $a, b \in M$ and $m, n \in \mathbb{N}$ such that $x = a : b$ and $r = m/n$, and define

$$rx = ma : nb.$$

It is easy to verify that the definition is independent of the choices of a, b and m, n. We leave the reader to verify the rest of the axioms for a set of ratios, except for the last two, which we shall now prove.

Lemma 1. *The ratios $X(M)$ satisfy the Archimedean axiom.*

Proof. We have to show that for all $x, y \in X(M)$, there exists n such that $x < ny$.

Let $x = a : b$ and $y = c : d$. By the Archimedean axiom for magnitudes, choose m such that $a < mb$, and choose n such that $md < nc$. Thus we have found multipliers $1, m$ such that $a < mb$ and $nc > md$. Therefore $a : b < nc : d$ by the definition of '$<$' for ratios.

Therefore $x < ny$, as required. □

Lemma 2. *The ratios $X(M)$ satisfy the new axiom.*

Proof. We have to show that given $x < y$, there exists $n \in \mathbb{N}$ such that $(n + 1)x < ny$. Let $x = a : b$ and $y = c : d$. By the definition of $x < y$, there exist $p, q \in \mathbb{N}$ such that: either

(i) $pa < qb$ and $pc \gtrsim qd$,

or

(ii) $pa \sim qb$ and $pc > qd$.

In case (i), there exists n such that $(n+1)pa < nqb$, by the new axiom for magnitudes. Therefore,

$$\begin{aligned}(n+1)x &= (n+1)a : b && \text{by definition of } \mathbb{Q}_+\text{-action}\\ &= (n+1)pa : pb\\ &< nqb : pb && \text{by choice of } n\\ &= nqd : pd\\ &\leqslant npc : pd && \text{since } pc \gtrsim qd\\ &= nc : d\\ &= ny.\end{aligned}$$

Therefore, $(n+1)x < ny$ by transitivity.

In case (ii), there exists n such that

$$(n+1)qd < npc$$

by the new axiom for magnitudes. Therefore,

$$\begin{aligned}(n+1)x &= (n+1)a : b\\ &= (n+1)pa : pb\\ &= (n+1)qb : pb && \text{since } pa \sim qb\\ &= (n+1)qd : pd\\ &< npc : pd && \text{by choice of } n\\ &= nc : d\\ &= ny,\end{aligned}$$

as required. □

Summarizing, we have the following theorem.

THEOREM 1. *Given a set M of magnitudes, then $X(M)$ satifies the axioms for a set of ratios, and hence also the axioms for a set of magnitudes. Therefore $X(X(M))$, the ratios of ratios of M, is well defined.*

3. *Additive structure*

DEFINITION OF ADDITIVE STRUCTURE. Let M be a set of magnitudes. We say that M has an *additive structure* if for all $a, b \in M$, there exists $a + b \in M$ satisfying the following axioms.

Commutative: $a + b \sim b + a$.

Associative: $(a + b) + c \sim a + (b + c)$.

Distributive: $n(a + b) \sim na + nb$;
$(m + n)a \sim ma + na$.

Equivalence: $a \sim b \Longrightarrow a + c \sim b + c$.

Order: $a < b \Longrightarrow a + c < b + c$.

Note that this does not make M into an additive group, because M may have not have a zero, nor negatives. However, M may also have subtraction.

Definition of subtraction. We say that M has *subtraction* if for all $a > b \in M$, there exists $a - b \in M$ such that

$$(a-b)+b \sim a.$$

One can then deduce that:

$$\begin{gathered}(a+b)-b \sim a;\\ \begin{array}{ccccc} a \sim b & \Longleftrightarrow & a+c \sim b+c & \Longleftrightarrow & a-c \sim b-c;\\ a < b & \Longleftrightarrow & a+c < b+c & \Longleftrightarrow & a-c < b-c; \end{array}\\ n(a-b) \sim na - nb.\end{gathered}$$

One can also deduce the four associative laws:

$$\begin{aligned}(a+b)+c &\sim a+(b+c);\\ (a+b)-c &\sim a+(b-c);\\ (a-b)+c &\sim a-(b-c);\\ (a-b)-c &\sim a-(b+c).\end{aligned}$$

Examples. The first six examples above of magnitudes have additive structure and subtraction: $\mathbb{N}, \mathbb{Q}_+, \mathbb{R}_+$, intervals, areas and volumes. The seventh example, of musical notes, however, has neither additive structure nor subtraction, as we explain below. Sets of ratios do not in general have an additive structure. In Section 7 we give an example that has additive structure but not subtraction.

Lemma 3. *If M is a set of magnitudes with additive structure and subtraction, then the Archimedean axiom implies the new axiom.*

Proof. Given $a < b$ in M, then there exists $b - a \in M$ by subtraction. By the Archimedean axiom choose n such that $a < n(b-a)$. Therefore $a < nb - na$, and thus

$$a + na < (nb - na) + na \sim nb,$$

by the subtraction axiom. Therefore $(n+1)a < nb$, as required. $\square$

Corollary. *The first six examples of sets of magnitudes obey the new axiom.*

Remark 3. The triviality of Lemma 3 explains why Euclid saw no need to draw attention to the new axiom. The latter comes into its own, however, when ratios have no additive structure.

Example (Example (vii) above: Musical notes). If a denotes a musical note, then its frequency is its natural measure, rather than the length of string being struck or plucked. Indeed, the note of a plucked string depends not only upon its length but also upon its tension and composition. Two strings of the same length and composition but different tension produce different notes, and strings of different lengths can give rise to the same note. The ordering of notes is determined by frequency, in other words by the chromatic scale. The $\mathbb{N}$-action is given by multiples of the frequency. Thus na is a note with frequency n times that of a. If a string gives rise to a note a, then na occurs naturally by stopping the string at $1/n$th of its length.

By contrast, there is no natural additive structure, nor is there any obvious physical realisation of one such that $a + a = 2a$. Hence musical notes are a set of magnitudes without additive structure or subtraction, but satisfying both the Archimedean axiom and the new axiom.

The ratios represent chords, such as an octave or a fifth, or major and minor thirds, each of which has its particular recognisable quality, independent of where it happens to lie in the chromatic scale.

4. *Overall perspective*

The main aim of *Book V* is to prove the elementary properties of ratios.

We now list the twenty-five propositions of *Book V*. The ten starred ones are called *additive propositions* because their statements involve the additive structure and their proofs follow at once from the latter. The other fifteen are called *non-additive propositions* because their statements do not involve the additive structure. Of these, Proposition 8 is circled because this is the only one for which Euclid uses the additive structure *in its proof*. We shall give an alternative proof in the next section.

4.1. *The twenty-five propositions of* Book V

Let $a, b, c, d, e, f, g, h \in M$ and $m, n \in \mathbb{N}$.

1^* $na + nb + \ldots \sim n(a + b + \ldots)$.

2^* $ma + na + \ldots \sim (m + n + \ldots)a$.

3 $m(na) \sim (mn)a$.

4 $a : b = c : d \implies ma : nb = mc : nd$.

5^* $na - nb \sim n(a - b)$.

6^* $ma - na \sim (m - n)a$.

7 $a \sim b \implies a : c = b : c$ and $c : a = c : b$.

8 $a < b \implies a : c < b : c$ and $c : a > c : b$.

9 $a : c = b : c \implies a \sim b$.

10 $a : c < b : c \implies a < b$.

11 $a : b = c : d = e : f \implies a : b = e : f$.

12^* $a : a' = b : b' = c : c' = \ldots \implies a : a' = (a + b + c + \ldots) : (a' + b' + c' + \ldots)$.

13 $a : b = c : d < e : f \implies a : b < e : f$.

14 $a : b = c : d \implies a <, \sim, > c$ as $b <, \sim, > d$.

15 $a : b = na : nb$.

16 $a : b = c : d \implies a : c = b : d$ (the alternando).

17^* $a : b = c : d \implies (a - b) : b = (c - d) : d$.

18* $a : b = c : d \quad \Longrightarrow \quad (a+b) : b = (c+d) : d.$

19* $a : b = c : d \quad \Longrightarrow \quad (a-c) : (b-d) = a : b.$

20 $a : b = d : e$ and $b : c = e : f \quad \Longrightarrow \quad a : c = d : f.$

21 $a : b = e : f$ and $b : c = d : e \quad \Longrightarrow \quad a <, \sim, > c$ as $d <, \sim, > f.$

22 $a : b = e : f$ and $b : c = f : g$ and $c : d = g : h \quad \Longrightarrow \quad a : d = e : h.$

23 $a : b = e : f$ and $b : c = d : e \quad \Longrightarrow \quad a : c = d : f.$

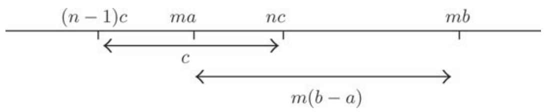
24* $a : c = d : f$ and $b : c = e : f \quad \Longrightarrow \quad (a+b) : c = (d+e) : f.$

25* $a : b = c : d$ and a the greatest $\quad \Longrightarrow \quad a + d > b + c.$

5. *Proofs of Euclid's Proposition* 8

Proposition 8. *$a < b \Longrightarrow a : c < b : c$ and $c : a > c : b$.*

Euclid's proof of Proposition 8 *using subtraction.*



By subtraction, $b - a$ exists.

By the Archimedean axiom, there exists m such that:

$$c < m(b-a) \tag{1}$$

and there exists n such that

$$ma < nc. \tag{2}$$

Choose n to be the least such, so that

$$(n-1)c \lesssim ma. \tag{3}$$

Add (1) and (3):

$$nc < mb. \tag{4}$$

We have found multipliers m, n in (2) and (4) such that

$$ma < nc \quad \text{and} \quad mb > nc.$$

Therefore $a : c < b : c$ by the definition of '$<$' for ratios. Similarly, $nc > ma$ and $nc < mb$, and so $c : a > c : b$. This completes Euclid's proof. □

Before we give the alternative proof using the new axiom, we shall need a couple of lemmas.

Lemma 4. *Given $a, b \in M$ and $m, n, m', n' \in \mathbb{N}$, then*

$$ma < nb \text{ and } n/m < n'/m' \quad \Longrightarrow \quad m'a < n'b.$$

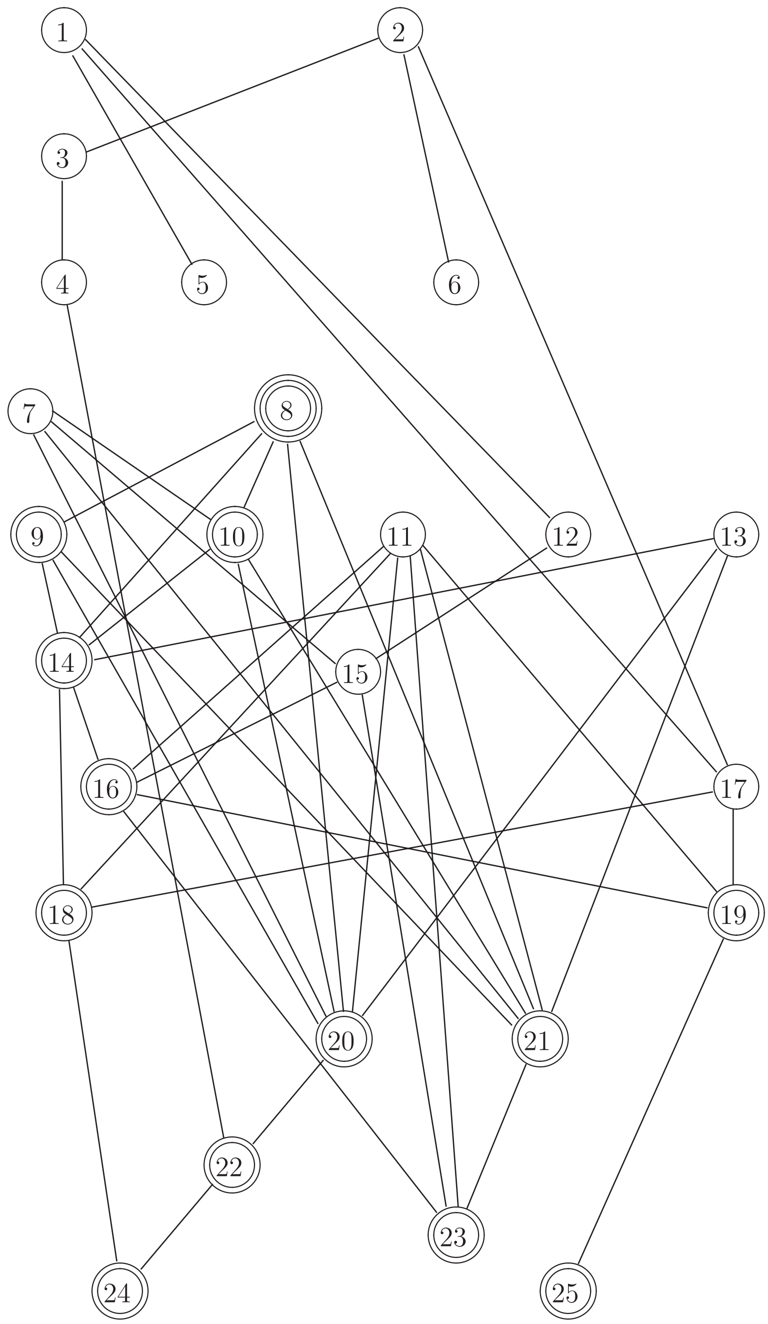


FIGURE 1. *The logical relationships between the propositions of* Book V. *A descending line from p to q indicates that $p \Longrightarrow q$; in other words p is used in the proof of q. The triple circle indicates Proposition* 8*, and the double circles indicate those propositions dependent on it.*

Proof. We see that

$$\begin{aligned}
m(m'a) &= m'(ma) && \text{by associativity and commutativity}\\
&< m'(nb) && \text{by hypothesis } ma < nb \text{ and axiom } a < b \Longrightarrow na < nb\\
&= (m'n)b && \text{by associativity}\\
&< (mn')b && \text{by hypothesis } m'n < mn'\text{, and axiom } p < q \Longrightarrow pa < qa\\
&= m(n'b) && \text{by associativity.}
\end{aligned}$$

Therefore $m'a < n'b$ by transitivity and the property that $na < nb \Longrightarrow a < b$. □

LEMMA 5. *We have $ma < nb$ and $m'a < n'b \Longrightarrow (m + m')a < (n + n')b$.*

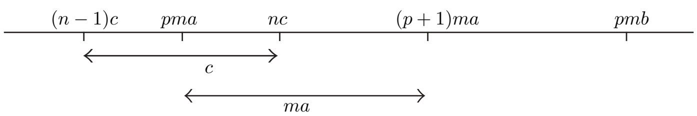
Proof. Suppose that $n/m < n'/m'$; otherwise the proof follows by symmetry. Thus

$$\frac{n}{m} < \frac{n + n'}{m + m'} < \frac{n'}{m'}.$$

Therefore $(m + m')a < (n + n')b$, by Lemma 4. □

REMARK 4. Notice that in Lemma 5 we have only used the addition of integers, and have not assumed any additive structure of magnitudes.

New proof of Proposition 8 using the new axiom. We have to show that $a < b \Longrightarrow a : c < b : c$.



By the new axiom, there exists p such that:

$$(p + 1)a < pb. \tag{5}$$

By the Archimedean axiom, there exists m such that

$$c < ma \tag{6}$$

and there exists n such that

$$pma < nc. \tag{7}$$

Choose n to be the least such, so that

$$(n - 1)c \lesssim pma. \tag{8}$$

Apply Lemma 5 to (8) and (6):

$$((n - 1) + 1)c < (pm + m)a.$$

Therefore

$$\begin{aligned}
nc &< (p + 1)ma\\
&= m((p + 1)a) && \text{by associativity and commutativity}\\
&< m(pb) && \text{by (5) and axiom } a < b \Longrightarrow na < nb\\
&= (pm)b && \text{by associativity and commutativity.}
\end{aligned}$$

Therefore

$$nc < (pm)b \quad \text{by transitivity.} \tag{9}$$

We have found multipliers pm, n in (7) and (9) such that

$$(pm)a < nc \quad \text{and} \quad (pm)b > nc.$$

Therefore $a : c < b : c$ by the definition of '$<$' for ratios. Similarly, $nc > (pm)a$ and $nc < (pm)b$ implies that $c : a > c : b$. This completes the new proof of Proposition 8. □

Euclid's proofs of all the other non-additive propositions follow from either Proposition 8 or the axioms for a set of magnitudes given in Section 2 above. Summarising, we have our next theorem.

THEOREM 2. *Given a set M of magnitudes, then the ratios $X(M)$ and the ratios of ratios $X(X(M))$ satisfy all the non-additive propositions of* Book V.

6. *Completion*

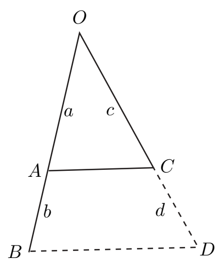
Define a set M of magnitudes to be *complete* if, for all $a, b, c \in M$, there exists $d \in M$ such that $a : b = c : d$.

Notice that this implies that given any three of a, b, c, d, then there exists the fourth. For instance, suppose that we are given a, b, d, then there exists c such that $b : a = d : c$. Hence $a : b = c : d$.

EXAMPLES. (i) $\mathbb{Q}_+$ and $\mathbb{R}_+$ are complete, for define $d = bc/a$.
(ii) $\mathbb{N}$ is not complete because if $a = 2$ and $b = c = 1$, then $d = 1/2 \notin \mathbb{N}$.

LEMMA 6. *Intervals are complete.*

Proof. Given intervals a, b, c draw two lines OAB, OC such that $OA = a$, $AB = b$, $OC = c$. Draw BD parallel to AC, and meeting OC in D. Let $d = CD$. Then $a : b = c : d$ by Euclid (*Book IV*, 2).



□

LEMMA 7. *If M is complete, then ratios of ratios exist in $X(M)$.*

Proof. Given $a, b, c, d \in M$, then there exists $e \in M$ such that $c : d = e : b$. Define

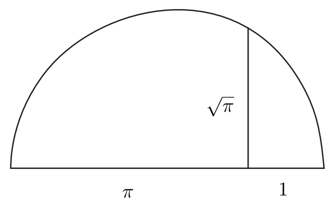
$$(a : b) : (c : d) = (a : b) : (e : b) = a : e.$$ □

LEMMA 8. *If M is complete and has an additive structure, then so has $X(M)$.*

Proof. Given $a, b, c, d \in M$, there exists $e \in M$ such that $c : d = e : b$. Define $(a : b) + (c : d) = (a : b) + (e : b) = (a + e) : b$. □

REMARK 5. One can show that if M is complete then so are $X(M)$ and $X(X(M))$. Thus it looks as though completeness is the answer to the problem of defining ratios of ratios. Unfortunately, it is not clear that all sets of magnitudes are complete.

COUNTEREXAMPLE 1. Let M be the set of regions in the plane. Let a be a square of side 1, and let b, c be circles of radius 1. Suppose that there exists d such that $a : b = c : d$. Let A, B, C, D denote the areas of a, b, c, d. Then $A = 1$, $B = C = \pi$ and $A/B = C/D$. Therefore $D = BC/A = \pi^2$. Therefore possible candidates for d are a square of side π or a circle of radius $\sqrt{\pi}$. The problem is to construct an interval of length either π or $\sqrt{\pi}$. Indeed, given either, then it is possible to construct the other, as follows.



However, it is not obvious how to construct either without the other. In fact, we run into the classical problem of squaring the circle: how to construct a square of area equal to that of a circle. If the circle has radius 1, then the square needs to have side $\sqrt{\pi}$.

Running into so many problems to solve one example indicates the size of the task for the ancient Greeks to show that M was complete. A modern proof that M is complete needs to use $\mathbb{R}$.

7. *The meaning of the new axiom*

The Archimedean axiom means there do not exist infinities, because any magnitude can be exceeded by a multiple of another. Similarly, the new axiom means there do not exist any infinitesimals, because the axiom

$$a < b \quad \Longrightarrow \quad \text{there exists } n \text{ such that } (n+1)a < nb$$

can be rewritten as

$$a : b < 1 \quad \Longrightarrow \quad \text{there exists } n \text{ such that } a : b < \frac{n}{n+1} < 1.$$

In other words, there exists a rational strictly between $a : b$ and 1. If ε were an infinitesimal such that $1 - \varepsilon < 1$, then $1 - \varepsilon$ would be too close to 1 to find a rational strictly between them, violating the new axiom.

Perhaps the reader will find it more convincing if we give a counterexample of a set of magnitudes that satisfies the Archimedean axiom but not the new axiom.

COUNTEREXAMPLE 2. Let $\mathbb{R}_+$ denote the positive reals. Let

$$\begin{aligned} M &= T\mathbb{R}_+ = \text{the tangent bundle of } \mathbb{R}_+ \\ &= \mathbb{R}_+ \times \mathbb{R} = \{(p, t);\ p \in \mathbb{R}_+,\ t \in \mathbb{R}\}. \end{aligned}$$

Here p represents a point on the line, and t a tangent to the line at p.

Order M lexicographically:

$$(p,t) < (p',t') \quad \text{if } p < p' \\ \text{or } p = p' \text{ and } t < t'.$$

Then the following statements hold.

(i) M satisfies the Archimedean axiom because the order is determined by the first term.

(ii) M does not satisfy the new axiom, for let $a = (1,0), b = (1,1)$. Then $a < b$, but for all n, $(n+1)a > nb$. In fact,

$$\begin{aligned} a : b &= 1_- \\ &= 1 \text{ minus an infinitesimal less than 1, but there does not exist a rational between.} \end{aligned}$$

(iii) M does not satisfy Proposition 8, for let $c = (1,-1)$. Then $a : c = b : c = 1_-$. Therefore

$$a < b \quad \text{but} \quad a : c \not< b : c.$$

(iv) M has an additive structure, but no subtraction because

$$a < b \quad \text{but} \quad b - a = (0,1) \notin M,$$

since $0 \notin \mathbb{R}_+$. Had Euclid used the new axiom, then we might have been saved from centuries of dodgy infinitesimals.

8. *Further developments*

Up to now we have written in the spirit of Euclid's mathematics; in this section we sketch how the theory can be pushed forward using modern ideas.

Given a set M of magnitudes, define inductively

$$\begin{gathered} X^1(M) = X(M) = \text{the set of ratios of } M; \\ X^2(M) = X(X^1(M)) = \text{the set of ratios of } X(M); \\ X^k(M) = X(X^{k-1}(M)). \end{gathered}$$

One can show that there are monomorphisms given by $x \mapsto x : 1$

$$X^1(M) \hookrightarrow X^2(M) \hookrightarrow X^3(M) \hookrightarrow \ldots \hookrightarrow X^k(M) \hookrightarrow \ldots,$$

where a *monomorphism* is a 1–1 map into, preserving the structures of order, unit and Q_+-action. Define

$$G = \bigcup_{k=1}^{\infty} X^k(M).$$

LEMMA 9. *G is an abelian multiplicative group.*

Sketch proof. Define

$$\begin{gathered} xy = x : (1 : y); \\ x^{-1} = 1 : x; \\ 1 = x : x. \end{gathered}$$

It is easy to verify that

$$\begin{aligned}(xy)z &= x(yz);\\ xy &= yx;\\ x1 &= x;\\ xx^{-1} &= 1.\end{aligned}$$

Lemma 10. *There exist monomorphisms $\mathbb{Q}_+ \hookrightarrow G \hookrightarrow \mathbb{R}_+$.*

Sketch proof. $Q_+ \hookrightarrow G$ by the $\mathbb{Q}_+$-action on 1.

To prove that $G \hookrightarrow \mathbb{R}_+$, we shall need to use the new axiom. Define $\mathbb{R}_+$ to be the set of Dedekind cuts of $\mathbb{Q}_+$ (see [4, Chapter 1]). Here, a Dedekind cut is partition of $\mathbb{Q}_+$ into two non-empty subsets (L, R) such that every L is less than every R, and L has no maximum. Define

$$\phi : G \hookrightarrow \mathbb{R}_+ \quad \text{by} \quad x \mapsto (L_x, R_x)$$

where

$$x = a : b, \quad L_x = \{q/p; pa > qb\}, \quad \text{and} \quad R_x = \{q/p; pa \leqslant qb\}.$$

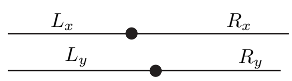
If $y = c : d$, we denote the corresponding image $\phi y = (L_y, R_y)$.

Then ϕ preserves 1 and $\mathbb{Q}_+$-action, and so to show that ϕ is a monomorphism it suffices to prove that $x < y \Longrightarrow \phi x < \phi y$.

From the definition of $x < y$, there exist p, q such that:

(i) $pa \leqslant qb$ and $pc > qd$, or
(ii) $pa < qb$ and $pc = qd$.

In case (i), $q/p \in R_x \cap L_y$, and so R_x meets L_y.



Hence $L_x \subsetneq L_y$. Therefore $\phi x < \phi y$, as required.

In case (ii), by the new axiom there exists n such that $(n+1)pa < nqb$. Therefore,

$$\frac{nq}{(n+1)p} \in R_x.$$

Also,

$$(n+1)pc > npc = nqd,$$

and so

$$\frac{nq}{(n+1)p} \in L_y.$$

Again, R_x meets Ly, and so $\phi x < \phi y$, as required. $\square$

Remark 6. Note that the new axiom was essential for this proof, because Lemma 10 fails for the tangent bundle of $\mathbb{R}_+$, the counterexample in § 7, which does not satisfy the new axiom.

9. *Equivalence classes*

I first began discussing Euclid *Book V* with David Fowler over 40 years ago. I complained to him that Euclid's definition of ratio in Definition 3 of *Book V* must be the vaguest and feeblest definition in the whole of Euclid. Furthermore, Euclid never actually uses it. Let me give the original Greek, and then Heath's 1908 translation of it into English [**3**].

Book V, Definition 3:

Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητα ποιὰ σχέσις.

'A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind.'

Compare this with my definition of ratio in Section 2 above:

$$a : b \text{ is the equivalence class of } (a, b)$$

under the equivalence relation given by Definition 5 (see Section 2 above). No-one complains about Definition 5 because this is the heart of the matter, the engine that drives the whole theory of proportion. If, then, Definition 5 is an equivalence relation between pairs of magnitudes, it follows logically that the ratios must be the corresponding equivalence classes. Therefore I suggested in [**5**] that perhaps it was a mistake to translate

ποιὰ σχέσις   as 'a sort of relation'

and that it might even be the technical term used by the Greek mathematicians of those days for what we now call 'an equivalence class'. David Fowler disagreed with me, and read several books on Greek mathematics, in which he found no evidence that the Greeks had invented equivalence relations. We argued gently about the matter for 40 years, both in private and in public, in seminars and before audiences of bemused undergraduates. The main benefit of our argument was that David himself became an eminent historian of mathematics, and even wrote books about ratios [**1**].

In our discussions we found ourselves following the traditional opposing roles of historian and mathematician. The historian thinks extrinsically in terms of the written evidence and adheres strictly to that data, whereas the mathematician thinks intrinsically in terms of the mathematics itself, which he freely rewrites in his own notation in order to better understand it and to speculate on what might have been passing through the mind of the ancient mathematician, without bothering to check the rest of the data.

Eventually I was persuaded by Sir John Boardman, an emeritus professor of Greek, that the words 'a sort of relation' may have enjoyed a more precise meaning in 1908 when Heath published his translation, but may now have become too casual, so that perhaps a better translation today might be 'a special relationship'. Meanwhile, I was persuaded by David Fowler that the idea of equivalence relations emerged during the 19th century, and that the idea of equivalence class was introduced by Richard Dedekind in 1871. In fact there is an unpublished paper [**2**] by David Fowler, giving all the details, that he completed shortly before his death in 2004.

However, in spite of being persuaded, I still think that Eudoxus and Euclid must have been thinking of a ratio as something like an equivalence class. The strongest linguistic evidence is Euclid's use of the word 'in'. Even today, we have inherited his rather surprising colloquial phrase of two pairs of magnitudes being '*in* the same ratio' rather than '*having* the same ratio'. In fact, the original Greek version of Definition 5 begins with the very words

Ἐν τῷ αὐτῷ λόγῳ ...   which translate into 'In the same ratio ...'.

Therefore Euclid must have been thinking that *ratio* was something that the two pairs could be *in* — in other words, an equivalence class. Consequently, I stick by my definition of *ratio* in Section 2 above.

Acknowledgements. I should like to end by acknowledging how much I enjoyed my discussions with David Fowler down the years, not only on Euclid but on all parts of mathematics and on every subject under the sun.

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