LMS Undergraduate Summer School Lecture Courses

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From the regular solids to quivers

by

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This course will be a whirlwind tour through representation theory, a major branch of modern algebra. We being by considering the symmetry groups of the regular solids, which leads naturally to the notion of a reflection group and its associated root system. The classification of these reflection groups gives us our first examples of quivers (= direct graphs). Though easy to define, well see that the representation theory associated to quivers is very rich. We will use quivers to illustrate the key concepts, ideas and problems that appear throughout representation theory. Coming full circle, the course will culminate with the beautiful theorem by Gabriel, classifying the quivers of finite type in terms of the roots systems of reflection groups. The ultimate goal of the course is to give students a glimpse of the beauty and unity of this field of research, which is today very active in the UK.

Recommended literature

H.S.M. Coxeter *Regular Polytopes.* Dover, 1973.

J.E. Humphreys *Reflection Groups and Coxeter Groups.* Cambridge Univ. Press, 1990.

Exercises: The Platonic solids

1. If the Schläfi symbol of the Platonic solid P is $\{p, q\}$, use Euler's formula $V - E + F = 2$ to show that

$$V = \frac{4p}{4 - (p-2)(q-2)}, \quad E = \frac{2pq}{4 - (p-2)(q-2)}, \quad F = \frac{4q}{4 - (p-2)(q-2)},$$

where V, E and F are the number of vertices, edges and faces respectively of P.

2. Recall from the first lecture that a *reflection* on $\mathbb{R}^n$ is a linear map $s : \mathbb{R}^n \to \mathbb{R}^n$ such that $\dim \mathrm{Fix}_{\mathbb{R}^n}(s) = n - 1$ and $s^2 = \mathrm{id}$.

   (a) Show that $\mathrm{Fix}_{\mathbb{R}^n}(s) = \mathrm{Ker}(\mathrm{id} - s)$.

   (b) Choose α such that $\mathrm{Fix}_{\mathbb{R}^n}(s) = H_\alpha$. Derive the formula

$$s_\alpha(x) = x - \frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha$$

   for a reflection.

   (c) Prove that s is diagonalizable. What are the eigenvalues of s_α?

   (d) Deduce that $\det(s_\alpha) = -1$.

   Hint: For part (c), if $x_1, \ldots, x_{n-1}$ is a basis of H_α, show that $x_1, \ldots, x_{n-1}, \alpha$ is a basis of $\mathbb{R}^n$ and calculate the action of s with respect to this basis.

3. There is a purely topological proof of the fact that there are are only five Platonic solids. The key topological fact is that Euler's formula holds: $V - E + F = 2$. Using this, together with the relations $pF = 2E = qV$, show that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{E}.$$

   Deduce that there are only five Platonic solids.

4. Using the fact that $g \in W(P)$ is a reflection if and only if it has one eigenvalue equal to -1 and two eigenvalues equal to 1, count the number of reflections in $W(\mathsf{H})$ and $W(\mathsf{D})$.

Exercises: Reflection groups and root systems

1. Let

$$E = \left\{ x = \sum_{i=1}^{n+1} x_i \epsilon_i \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0 \right\},$$

where $\{\epsilon_1, \ldots, \epsilon_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$ with $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. Let $R = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n+1\}$.

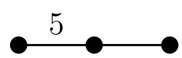
   (i) Show that R is a *crystallographic* root system.

   (ii) Construct two different sets of simple roots for R.

   (iii) By considering the action of the reflections $s_{\epsilon_i - \epsilon_j}$ on the basis $\{\epsilon_1, \ldots, \epsilon_{n+1}\}$ of $\mathbb{R}^{n+1}$, show that the Weyl group of R is isomorphic to $\mathfrak{S}_{n+1}$.

2. Show that the symmetric matrix

$$A = \begin{pmatrix} 1 & -\cos\frac{\pi}{5} & 0 \\ -\cos\frac{\pi}{5} & 1 & -\cos\frac{\pi}{3} \\ 0 & -\cos\frac{\pi}{3} & 1 \end{pmatrix}$$

corresponding to the Coxeter graph 

of type H_3 is positive definite. What is the determinant of A? Hint: recall that $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2}$.

3. The angle between roots in a crystallographic reflection groups. Recall the following table in section 2.5 of the lecture notes. The only possible values of $\langle \alpha, \beta \rangle$ are:

$\langle \beta, \alpha \rangle$	$\langle \alpha, \beta \rangle$	θ
0	0	$\frac{\pi}{2}$
1	1	$(\star)$
-1	-1	$\frac{2\pi}{3}$
1	2	$(\star\star)$
-1	-2	$\frac{3\pi}{4}$
1	3	$\frac{\pi}{6}$
-1	-3	$(\star\star\star)$

   (a) What are the angles θ in $(\star)$, $(\star\star)$ and $(\star\star\star)$?

   (b) What about $\langle \beta, \alpha \rangle = \langle \alpha, \beta \rangle = \pm 2$?

(c) Let $\alpha, \beta \in \mathbb{R}^n$. Show that $s_\beta s_\alpha$ is a rotation of $\mathbb{R}^n$. Hint: decompose $\mathbb{R}^n = \mathbb{R}\{\alpha, \beta\} \oplus H_\alpha \cap H_\beta$ and consider $s_\beta s_\alpha$ acting on $\mathbb{R}\{\alpha, \beta\}$. If e_1, e_2 is an orthonormal basis of $\mathbb{R}^2$, write out s_α and s_β explicit.

4. The *hypercube* H_n is the n-dimensional analogue of the square ($n = 2$), or cube ($n = 3$). Concretely, we can realize H_n in $\mathbb{R}^n$ as the set of points

$$\mathsf{H}_n = \{v \in \mathbb{R}^n \mid -1 \leq v_i \leq 1 \; i = 1, \dots, n\}.$$

The group of symmetries of H_n is denoted B_n. It is called the *hyperoctahedral group.*

(i) How many vertices does the H_n have? How about edges, or faces?

(ii) The $(n-1)$-dimensional faces of H_n are the copies of H_{n-1} given by $\{v \in \mathsf{H}_n \mid v_i = 0\}$. Using the fact that B_n permutes these $(n-1)$-dimensional faces, show that B_n permutes the set $\{e_i^\pm \mid i = 1, \dots, n\}$, where

$$e_i^\pm = (0, \dots, 0, \pm 1, 0, \dots, 0).$$

(iii) Deduce that w is a sign permutation matrix i.e. a matrix where each row has only one non-zero entry which is either a 1 or -1, and similarly for the columns.

(iv) What is the order of the group B_n?

Exercises: Quivers

1. A *homomorphism* between representations. Let $M = \{(\mathbb{C}^{v_i}, \varphi_\alpha)\}$ and $N = \{(\mathbb{C}^{w_i}, \psi_\alpha)\}$ be representations of a quiver Q. Then a homomorphism $\mathbf{f} : M \to N$ is a collection of linear maps $f_i \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$ for each $i \in Q_0$ such that the diagrams

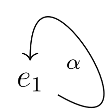
$$\begin{array}{ccc} \mathbb{C}^{v_{t(\alpha)}} & \xrightarrow{\varphi_\alpha} & \mathbb{C}^{v_{h(\alpha)}} \\ \downarrow{\scriptstyle f_{t(\alpha)}} & & \downarrow{\scriptstyle f_{h(\alpha)}} \\ \mathbb{C}^{w_{t(\alpha)}} & \xrightarrow{\psi_\alpha} & \mathbb{C}^{w_{h(\alpha)}} \end{array}$$

commute for all $\alpha \in Q_1$. The space of all homomorphisms from M to N is denoted $\mathrm{Hom}_Q(M, N)$.

   (a) Consider the representations

$$M : \quad \mathbb{C}^2 \overset{(a,b)}{\underset{(c,d)}{\rightrightarrows}} \mathbb{C} \qquad\qquad N : \quad \mathbb{C} \overset{x}{\underset{y}{\rightrightarrows}} \mathbb{C}$$

   where $a, b, c, d, x, y \in \mathbb{C}$. If $(a, b) = (2, 1)$, $(c, d) = (6, 3)$, $x = 1$ and $y = 3$, construct a non-zero homomorphism $\mathbf{f} : M \to N$. Are there any homomorphisms $\mathbf{f} : M \to N$ when $(a, b) = (2, 2)$, $(c, d) = (6, 4)$, $x = 2$ and $y = 2$? In general, what conditions do a, b, c, d, x and y need to satisfy for $\mathrm{Hom}_Q(M, N)$ to be non-zero? What is the dimension of $\mathrm{Hom}_Q(M, N)$ in this case?

   (b) Recall that the representations of the quiver  are simply pairs $(\mathbb{C}^n, A)$,

   where $A : \mathbb{C}^n \to \mathbb{C}^n$ is an $n \times n$ matrix. If $M = (\mathbb{C}^n, A)$, show that $\mathrm{Hom}_Q(M, M) = \{B : \mathbb{C}^n \to \mathbb{C}^n \,|[A, B] = 0\}$, where $[A, B] := AB - BA$ is the *commutator* of A and B.

2. Let Q be a quiver. Recall that, for each $i \in Q_0$, we have defined the representation $E(i)$ of Q.

   (a) Show that the representation $E(i)$ is simple.

   (b) If Q has no oriented cycles, show that every simple representation equals $E(i)$ for some $i \in Q_0$.

(c) Consider the quiver $e_1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} e_2$. Show that the representation $\mathbb{C} \underset{2}{\overset{3}{\rightleftarrows}} \mathbb{C}$ is simple.

3. Let Q be the quiver

$$\begin{array}{ccccc} & & e_2 & & \\ & & \downarrow{\scriptstyle \alpha} & & \\ e_1 & \xrightarrow{\beta} & e_5 & \xrightarrow{\gamma} & e_3 \\ & & \downarrow{\scriptstyle \delta} & & \\ & & e_4 & & \end{array}$$

Write down the basis of paths for the path algebra $\mathbb{C}Q$. What is $\dim \mathbb{C}Q$?

4. Let Q be the quiver $e_1 \xrightarrow{\alpha} e_2 \xrightarrow{\beta} e_3$ and let

$$A = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \;\middle|\; a, b, c, d, e, f \in \mathbb{C} \right\}$$

be the algebra of upper triangular 3×3 matrices, where multiplication is just the usual matrix multiplication. Construct an explicit isomorphism of algebras $\mathbb{C}Q \xrightarrow{\sim} A$.

Exercises: Gabriel's Theorem

1. You'll notice that the positive definite Euler graphs are precisely the positive definite Coxeter graphs that are *simply laced* i.e. have at most one edge between any two vertices. Let $(-,-)_C$, resp. $(-,-)_E$, be the Coxeter form, resp. the Euler form, associated to a graph Γ.

   (a) Show that if Γ is simply laced then $(-,-)_E = 2(-,-)_C$.

   (b) If Γ is not simply laced, show that there is no $\lambda \in \mathbb{R}$ such that $(-,-)_E = \lambda(-,-)_C$.

   (c) Show that the symmetric matrix

   $$\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$$

   corresponding to the Euler graph $\bullet\!\overset{m}{\longrightarrow\!\!\!\!\!-}\!\bullet$ is positive definite if and only if $m = 1$. When is it positive semi-definite?

   (d) By considering the subgraphs $\bullet\!\overset{m}{\longrightarrow\!\!\!\!\!-}\!\bullet$ with $m > 1$ of Γ, show that a non-simply laced Euler graph is not positive definite.

   (e) Deduce Theorem 4.8 from Theorem 2.18.

2. Let $i \in Q_0$ be a sink. Show that $S_i^+(E(i)) = 0$.

3. Consider the representation M given by

   $$\begin{array}{ccccc} & & \mathbb{C} & & \\ & & {\scriptstyle (1,0)}\uparrow & & \\ \mathbb{C} & \xleftarrow[(1,2)]{} & \mathbb{C}^2 & \xrightarrow{(0,1)} & \mathbb{C} \\ & & \downarrow{\scriptstyle (1,1)} & & \\ & & \mathbb{C} & & \end{array}$$

   If we label the central vertex by i, what is $S_i^-(M)$?

4. Let Q be the quiver $e_1 \xrightarrow{\alpha} e_2 \xleftarrow[\beta]{} e_3$ of type A_3. The corresponding root system, with reflection group $\mathfrak{S}_4$ was considered in the first exercise on reflection groups and root systems. Thus, the positive roots are

   $$R^+ = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3 - \epsilon_4, \epsilon_1 - \epsilon_3, \epsilon_2 - \epsilon_4, \epsilon_1 - \epsilon_4\},$$

which, under the identification $e_i \mapsto \epsilon_i - \epsilon_{i+1}$, corresponds to

$$R^+ = \{e_1, e_2, e_3, e_1 + e_2, e_2 + e_3, e_1 + e_2 + e_3\}.$$

For each of the above dimension vectors construct an explicit indecomposable representation of Q.

Introduction to the theory of complex networks

by

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Complex networks are increasingly popular and useful as representations of complex systems that consist of many interacting variables. They now appear as the lingua franca in many disciplines, ranging from engineering, physics and computer science, via economics, financial markets and the social sciences, to biology and medicine. The nodes (or vertices) of the networks are taken to represent the variables of the dynamical system, and links (or bonds) tell us which pairs of nodes share a direct connection. The rationale behind network science is that many features of complex many-variable systems can be deduced already from the topology of the network of pair-interactions alone.

This short course gives a gentle introduction to the field of complex networks and graphs. It consists of three parts. In the first part we give examples of networks and graphs in different disciplines, and introduce the definitions and terminology that are used to characterise and quantify their topological features. This includes adjacency matrices, degrees, degree distributions, clustering coefficients, modularity, and path length statistics. In the second part we illustrate the connection between dynamical processes that are defined for variables on the nodes of networks and the eigenvalue spectra of their adjacency matrices and Laplacian matrices, and we establish several properties of these spectra. The third part is devoted to random graph ensembles, focusing mainly on the Erdös-Rènyi ensemble, and on phase transitions in such ensembles.

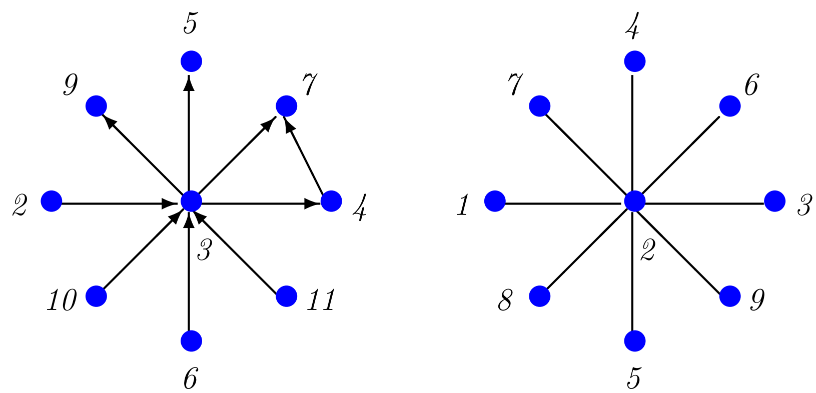
Recommended literature:

MEJ Newman *Networks: An Introduction.* Oxford UP, 2010.

B Bollobas *Random Graphs.* Cambridge UP, 2001.

Introduction to the Theory of Complex Networks
EXERCISES

(i) Which if the two graphs on the right is simple? Which is directed? Give for each graph the vertex set V and the edge set E.



(ii) Calculate the adjacency matrices for the two graphs above, upon relabelling the nodes of the first graph such that its vertex set becomes $V = \{1, \ldots, 9\}$.

(iii) Calculate the clustering coefficients for all nodes in the second of the above graphs. Why would we not calculate them for the first graph?

(iv) Calculate the closeness centrality and the betweenness centrality of nodes $i = 2$ and $i = 3$ in the second graph of (i).

(v) Prove that the *average* in-degree $\overline{k}^{\text{in}}(\boldsymbol{A}) = \frac{1}{N}\sum_i k_i^{\text{in}}(\boldsymbol{A})$ and the *average* out-degree $\overline{k}^{\text{out}}(\boldsymbol{A}) = \frac{1}{N}\sum_i k_i^{\text{out}}(\boldsymbol{A})$ of *any* graph are always identical. Show that in simple nondirected graphs the number of links is $L = \frac{1}{2}N\bar{k}(\boldsymbol{A})$, where $\bar{k}(\boldsymbol{A}) = N^{-1}\sum_{i\leq N} k_i(\boldsymbol{A})$ is the average degree.

(vi) Prove the following general bounds for the modularity: $-\frac{1}{2} \leq Q(\boldsymbol{A}) \leq \frac{1}{2}$.

(vii) Assign the following module labels to the nodes of the right graph in exercise (i): $x_1 = x_2 = 1$, $x_3 = x_4 = x_5 = 2$. Calculate the graph's modularity $Q(\boldsymbol{A})$.

(viii) Calculate the spectrum $\varrho(\mu|\boldsymbol{A})$ of the second graph in (i). Use your result to calculate the average degree, and to prove that this graph has no closed paths of odd length.

(ix) Show how for regular N-node graphs one can express the Laplacian eigenvalue spectrum $\varrho_{\text{Lap}}(\mu|\boldsymbol{A})$ in terms of the adjacency matrix eigenvalue spectrum $\varrho(\mu|\boldsymbol{A})$.

(x) For the Erdös-Rènyi model we know that $\langle\bar{k}(\boldsymbol{A})\rangle = p^\star(N-1)$. Calculate $\langle\bar{k}^2(\boldsymbol{A})\rangle$. Calculate the variance $\sigma_{\bar{k}}^2 = \langle\bar{k}^2(\boldsymbol{A})\rangle - \langle\bar{k}(\boldsymbol{A})\rangle^2$ in the finite connectivity regime, and express it in terms of $\langle k\rangle$ for $N \to \infty$. What can you conclude from the result?

(xi) Let $\alpha \in [0, 1]$ and $q_1, q_2 \in \mathbb{N}$. Calculate the generating function $G(x)$ for the following degree distribution: $p(k) = \alpha\delta_{k,q_1} + (1-\alpha)\mathrm{e}^{-q_2}q_2^k/k!$.

(xii) Confirm that the three generating functions for regular, Poissonnian, and exponential random graphs all obey: $G(0) = p(0)$, $G(1) = 1$, and $\lim_{x\to1} x\frac{\mathrm{d}}{\mathrm{d}x}G(x) = \langle k\rangle$. Calculate expressions for $\langle k^2\rangle$ from the three generating functions.

(xiii) Prove that $\langle k^2\rangle \geq \langle k\rangle$ for all graphs, with equality if and only if $p(k) = 0$ for all $k > 1$.

(xiv) Construct a large 2-regular graph, i.e. one with $p(k) = \delta_{k,2}$ and large N, that does not have a giant component. Prove your claim.

A complex life

by

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"*But think of Adam and Eve like an imaginary number, like the square root of minus one: you can never see any concrete proof that it exists, but if you include it in your equations, you can calculate all manner of things that couldn't be imagined without it.*"

The Golden Compass, Philip Pullman

Imaginary, or complex, numbers have long fascinated not just mathematicians but the public at large; it is bemusing and intriguing that an "imagined" abstraction can have real-life utility. In this short lecture series I will give evidence, drawn mainly from my own research interests, that complex analysis is an indispensable – and powerful – mathematical tool with perennial relevance to modern day applications in science and engineering. I'll have something to say about lotus leaves, ketchup, optical fibres and micro-robots, among other things.

The mathematics and physics of random matrices

by

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Lectures at the LMS Undergraduate Summer School
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What are the eigenvalues of a matrix with random entries? This type of question appears today in various areas of mathematics and physics. We will make the question more precise in specific examples and explain the relation to orthogonal polynomials, electrostatic equilibrium and conformal mappings.

Recommended literature

M. L. Mehta *Random Matrices.* San Diego, Academic Press, 1991.

P. A. Deift *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach.* Courant Lecture Notes in Mathematics, Vol 3, NY, 1999.

G. W. Anderson, A. Guionnet, O. Zeitouni *An introduction to random matrices.* Cambridge University Press, 2010.

THE MATHEMATICS AND PHYSICS OF RANDOM MATRICES

Exercise sheet and literature

We'll mostly discuss (2), (4), (6) in class. Problem (1) is a teaser and the other exercises are supposed to provide background material.

(1) Build a big square matrix, say 400 by 400, with zeros and ones chosen by flipping a coin for each entry. Plot its eigenvalues fuin the complex planes. Where are they? (I suggest that you use a computer to both flip the coin and compute the eigenvalues)

(2) Let $S_2(\mathbb{R}) = \{X = (x_{ij}) \in M_2(\mathbb{R}) \,|\, x_{12} = x_{21}\}$ be the space of real symmetric 2 by 2 matrices. Show that if $f\colon S_2(\mathbb{R}) \to \mathbb{R}$ is continuous and invariant under conjugation by orthogonal matrices,

$$\int_{S_2(\mathbb{R})} f(X)dx_{11}dx_{12}dx_{22} = \pi \int_{\lambda_1 \geq \lambda_2} f\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (\lambda_1 - \lambda_2)d\lambda_1 d\lambda_2.$$

(3) A complex $n \times n$ matrix A is called *normal* if $AA^* = A^*A$ (A^* is the adjoint matrix, obtained by transposing A and taking complex conjugated entries). TFAE: (i) A is normal (ii) $A = U\Lambda U^*$ for some unitary matrix U and diagonal matrix Λ (iii) A has an orthonormal basis of eigenvectors (iv) $A = X + iY$ for some hermitian matrices X, Y such that $XY = YX$.

(4) Let $f_N(y) = \sum_{k=0}^{N} y^k/k!e^{-y}$. Show that

$$\lim_{N\to\infty} f_N(Nx) = \begin{cases} 1 & \text{if } 0 \leq x < 1, \\ 0, & \text{if } x > 1. \end{cases}$$

Hint: show that $f_N(y)$ can be written as $1 - \int_0^y e^{-t}t^N/N!dt$ and google "Laplace's method" (while you're at it, Wikipedia needs your help on this topic).

(5) The purpose of this exercise is to translate the statement of the Gauss divergence theorem

$$\int_D (\partial_x v_1 + \partial_y v_2)dxdy = \int_{\partial D} v \cdot n\, ds$$

for a compact region $D \subset \mathbb{R}^2$ with smooth boundary ∂D to the language of differential forms.

(a) The differential of a function $f(x, y)$ is the 1-form $\partial_x f dx + \partial_y f dy$. Show that $d(fg) = fdg + gdf$ where the product of a function by a 1-form is defined as $f(pdx + qdy) = fpdx + fqdy$.

(b) The differential of a 1-form $\alpha = p(x, y)dx + q(x, y)dy$ is defined as $d\alpha = (\partial_x q - \partial_y p)dxdy$. Deduce the Stokes formula

$$\int_D d\alpha = \int_{\partial D} \alpha$$

from the divergence theorem. Discuss the question of orientation of the line integral on the right.

(c) Identify $\mathbb{R}^2$ with $\mathbb{C}$ via $(x, y) \mapsto z = x + iy$. Show that $df = \partial_z f dz + \partial_{\bar{z}} f d\bar{z}$ and $d(pdz + qd\bar{z}) = (\partial_z q - \partial_{\bar{z}} p) dz d\bar{z}$ with $dz d\bar{z} = -2i dx dy$. Here $\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$.

(d) If f is homolorphic then $df(z) = f'(z)dz$ where f' is the complex derivative of f.

(6) The Navier–Stokes equations for the velocity field $u(t, x, y, z) \in \mathbb{R}^3$ and pressure field $p(t, x, y, z) \in \mathbb{R}$ of a viscous incompressible fluid in the absence of external forces are

$$\rho(\partial_t u + (u \cdot \nabla)u) = \mu \Delta u - \nabla p, \quad \mathrm{div} u = 0.$$

The density ρ and the viscosity μ are positive constant parameters. The differential operators $u \cdot \nabla = u_1 \partial_x + u_2 \partial_y + u_3 \partial_z$ and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ act on each component of u.

(a) Find the solution in the space between plates at $z = \pm b/2$ such that (i) $u = u(z)$ only depends on z, (ii) u obeys the no slip boundary conditions $u(\pm b/2) = 0$, (iii) the pressure gradient ∇p is a constant vector perpendicular to the z-axis. The solution is unique up to adding a constant to p.

(b) Show for this solution, the average velocity $v = \frac{1}{b} \int_{-b/2}^{b/2} u(z) dz$ is proportional to the pressure gradient:

$$v = -\frac{b^2}{12\mu} \nabla p.$$

References

[1] M. L. Mehta. Random Matrices. San Diego, Academic Press, 1991

[2] P. A. Deift, Orthogonal polynomials and random matrices: a Riemann-Hilbert approach. Courant Lecture Notes in Mathematics, Vol 3, New York: Courant Institute of Mathematical Sciences, 1999

[3] G. W. Anderson, A. Guionnet, O. Zeitouni, An introduction to random matrices, Cambridge University Press, 2010

Introduction to tropical geometry.

by

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Algebraic geometry is the study of solution sets to polynomial equations. Typically, one works over algebraically closed fields such as the complex numbers, and the first interesting examples are algebraic plane curves, i.e., curves defined by an equation of the form $f(x, y) = 0$, where x and y can take values in the complex numbers. Here f is a polynomial in two variables.

The solution sets to such equations are in general two real-dimensional objects sitting inside a real four-dimensional vector space, and hence tend to be difficult to visualize. However, it is easier to draw a two-dimensional picture by considering the "amoeba" of the curve, i.e., the image of the curve under the map $\mathbb{C}^2 \to \mathbb{R}^2$ given by taking absolute value. For certain curves, it is easy to understand what the amoeba looks like. This leads us to piecewise linear, or limiting, approximations to amoebas known as "tropical curves."

Tropical geometry is the study of these limiting objects, and can viewed as a form of algebraic geometry over not the field $\mathbb{C}$ but the so-called "tropical semi-ring" or "max-plus semi-ring". This semi-ring, as a set, is the set of real numbers, but addition is replaced with maximum and multiplication by addition. Thanks to work of Mikhalkin and others, it is now understood that many classical algebraic geometry results have tropical analogues which are purely combinatorial in nature.

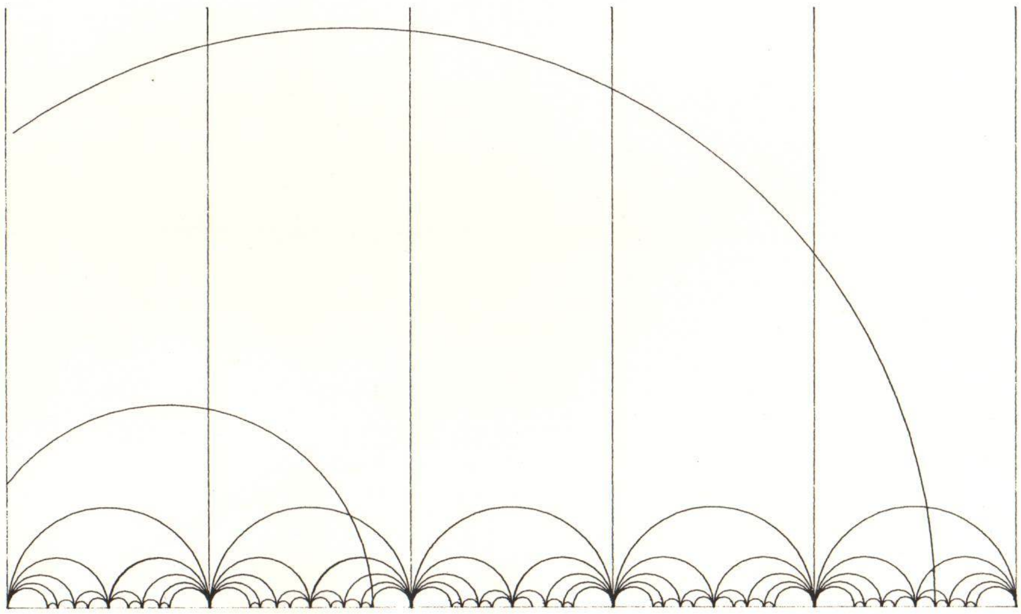
In this series of lectures, I will explore the motivation for tropical geometry and state and prove some elementary results, as well as discuss their application to curve-counting.

Continued Fractions and Hyperbolic Geometry

Caroline Series

Loughborough LMS Summer School

July 2015



Outline

Why is it that 22/7 and 355/113 are chosen as good approximations to π? In fact 355/113 = 3 + 1/(7 + 1/16) approximates π to six decimal places. They are examples of continued fractions, which are used to get 'best approximations' to an irrational number for a given upper bound on the denominator, so-called Diophantine approximation.

There is a beautiful connection between continued fractions and the famous tiling of the hyperbolic (non-Euclidean) plane shown in the figure above. It is called the Farey tessellation and its hyperbolic symmetries are the 2x2 matrices with integer coefficients and determinant one, important in number theory. We shall use the Farey tessellation to learn about both continued fractions and hyperbolic geometry, leading to geometrical proofs of some classical results about Diophantine approximation.

Lecture 1 We describe the Farey tessellation $\mathcal{F}$ and give a very quick introduction to the basic facts we need from hyperbolic geometry, using the upper half plane model.

Lecture 2 We introduce continued fractions and explain the relationship between continued fractions and $\mathcal{F}$.

Lecture 3 We use $\mathcal{F}$ to visualise some classical results about continued fractions and outline a few of the many applications and further developments.

Everything needed about continued fractions and hyperbolic geometry will be explained in the lectures, but to prepare in advance you could look at any of the many texts on these subjects. Here are a few sources:

G. H. Hardy and E. M. Wright. *The Theory of Numbers.* Oxford University Press, Many editions.

A. Ya. Khinchin *Continued Fractions.* University of Chicago Press, 1935.

C. Series. *Hyperbolic geometry notes MA448.* Unpublished lecture notes, available at `homepages.warwick.ac.uk/~masbb/`

For an introduction to the Farey tessellation and continued fractions from a slightly different viewpoint see

A. Hatcher. *Toplogy of Numbers.* Unpublished draft book, available at `www.math.cornell.edu/~hatcher/TN/TNpage.html`

1 The Farey Tessellation and the hyperbolic plane

Fractions $p/q, r/s \in \mathbb{Q}$ are called *neighbours* if $|ps - rq| = 1$. Their *Farey sum*, denoted $p/q \oplus_F r/s$, is defined to be $(p + r)/(q + s)$. Note that if $p/q < r/s$ are neighbours, then so are $p/q < p/q \oplus_F r/s$ and $p/q \oplus_F r/s < r/s$. Figure 1, drawn in the complex plane, is formed by the following procedure:

- Draw vertical lines from n to ∞ at each integer point $n \in \mathbb{R}$. Label these points $n/1$. Note that for each $n \in \mathbb{Z}$, the pair $(n/1, (n + 1)/1)$ are neighbours.
- Join each adjacent pair $(n/1, (n + 1)/1)$ by a semicircle with its centre on $\mathbb{R}$.
- Mark the point $n/1 \oplus_F (n+1)/1 = (2n+1)/2$. Join the adjacent neighbours $n/1, (2n+1)/2$ and $(2n + 1)/2, (n + 1)/1$ by semicircles centred on $\mathbb{R}$.
- Inductively, suppose that $p/q < r/s$ are Farey neighbours joined by an arc. Join p/q to $(p + r)/(q + s)$ and $(p + r)/(q + s)$ to r/s by semicircles.
- Continue in this way.

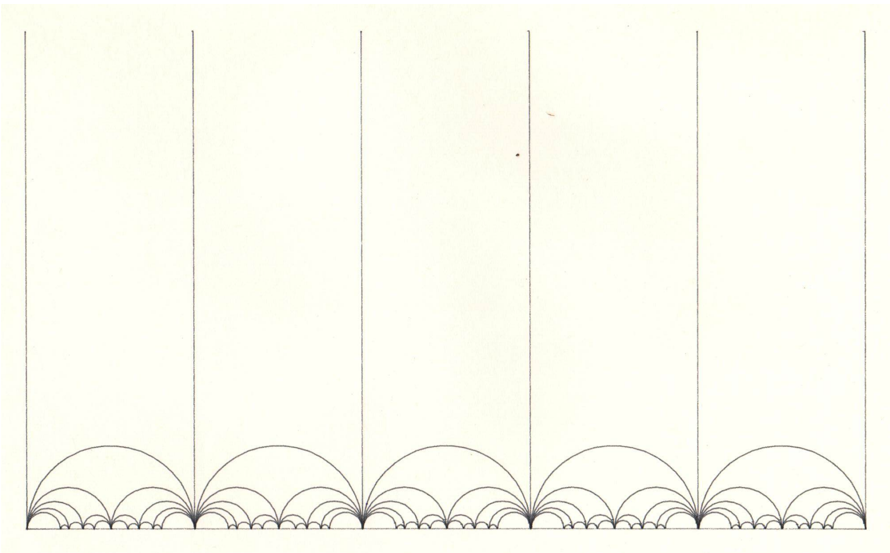


Figure 1: The Farey Tessellation

Exercise 1.1. *Check by induction that if $p/q, r/s$ are joined by an arc of $\mathcal{F}$ then* $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ *has determinant* ± 1.

The Farey tessellation is a tessellation or tiling of the *hyperbolic plane*. This means there is a basic figure, a so-called *ideal triangle*, whose images under some group of symmetries cover the hyperbolic plane without overlaps.

To understand this we need a bit of background on hyperbolic geometry. Everything we shall use is worked through in detail in the first few chapters of [10], but we explain what we need briefly here. Hyperbolic geometry originated as geometry in which Euclid's parallel postulate fails. It is the geometry of space with constant curvature -1. All we need to know is that 2-dimensional hyperbolic geometry can be *modelled* as the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ with the *metric* $ds^2 = (dx^2 + dy^2)/y^2$, where $z = x + iy$. What this means is that to find the length of an arc γ joining points A, B we have to integrate: $\ell(\gamma) = \int_\gamma ds = \int_\gamma \sqrt{dx^2 + dy^2}/y$ and $d_{\mathbb{H}}(A, B) = \inf_\gamma \ell(\gamma)$.

Here is an example. Let $A = ai$ and $B = bi$ so that A, B are on the imaginary axis $\mathbb{I}$, and assume $b > a$. Let γ be any arc joining A to B. Then

$$\ell(\gamma) = \int_\gamma ds = \int_\gamma \sqrt{dx^2 + dy^2}/y \geq \int_\gamma dy/y = \int_{y=a}^{y=b} dy/y = \log b/a.$$

Moreover if we take γ_0 to be the vertical path from A to B then $\ell(\gamma_0) = \log b/a$. Hence $d_{\mathbb{H}}(ai, bi) = \log b/a$. Note that this shows that the vertical path γ_0 is a shortest distance path, otherwise called a *geodesic* or a *hyperbolic line*.

The boundary at infinity The above formula shows that $d_{\mathbb{H}}(i, ti) \to \infty$ as $t \to 0$. Thus the real axis is at infinite distance from a point in $\mathbb{H}$. Notice that the real axis $\mathbb{R}$ is *not* included in $\mathbb{H}$. Clearly the point ∞ is also at infinite distance from any point in $\mathbb{H}$. We view $\mathbb{R} \cup \infty$ as a circle, known as the *boundary (or circle) at infinity*.

1.1 Isometries of $\mathbb{H}$

To understand a geometry and its tilings we need to understand its *isometries*, that is, its distance preserving maps. The isometries of $\mathbb{H}$ have a very nice description in terms of the group $SL(2, \mathbb{R})$. This is the group of 2×2 matrices with real entries and determinant 1, i.e. $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1\right\}$. $SL(2, \mathbb{R})$ acts on $\mathbb{H}$ in the following way. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$ and $z \in \mathbb{H}$. Then $T(z) = (az + b)/cz + d)$. By convention, $T(\infty) = a/c$ and $T(-d/c) = \infty$.

Exercise 1.2. *Show that:*

a. if $\Im z > 0$ *then* $\Im(az + b)/cz + d) > 0$.

b. T *maps the circle at infinity to itself.*

c. *if* $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $T' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ *then* $T'(T(z)) = (T'T)(z)$*, where* $T'T$ *is the matrix product of* T' *with* T *and* $T'(T(z))$ *is the image of* $T(z)$ *under* T'*.*

d. *if* $(az+b)/(cz+d) \equiv z$ *then* $a = d = \pm 1, b = c = 0$*.*

Exercise 1.2 (c) shows that to compose maps we simply need to multiply matrices. (d) shows that the group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\pm \mathrm{Id}$ (where $\mathrm{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$) acts freely on $\mathbb{H}$, that is, if $T(z) = z$ then $T = \mathrm{id}$ as an element of $PSL(2,\mathbb{R})$. Where it won't lead to confusion, we often use $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to represent a transformation in $PSL(2,\mathbb{R})$.

Proposition 1.1. *$PSL(2,\mathbb{R})$ acts by isometries on $\mathbb{H}$. In other words, if $T \in PSL(2,\mathbb{R})$, then $d_{\mathbb{H}}(T(P), T(Q)) = d_{\mathbb{H}}(P,Q)$ for any $P,Q \in \mathbb{H}$.*

Proof. To abbreviate, write $|dz| = \sqrt{dx^2 + dy^2}$. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and let $w = T(z) = (az+b)/cz+d)$. We claim that $|dw|/\Im w = |dz|/\Im z$ and consequently $\int_\gamma |dz| = \int_{T(\gamma)} |dw|$.

Exercise 1.3. *Finish the proof!*

Linear fractional transformations

A mapping of the form $z \mapsto (az+b)/cz+d)$, where $a,b,c,d \in \mathbb{C}$ and $ad - bc \neq 0$ is called a *linear fractional transformation* or *Möbius map*. Möbius maps carry circles to circles and preserve angles. Here 'circle' is interpreted to mean either an ordinary circle or a line through infinity. For more details and a proof see [10] Chapter 1.

Exercise 1.4. a. *Show that under the action of $PSL(2,\mathbb{R})$, a vertical line in $\mathbb{H}$ is carried either to another vertical line or to a semicircle centred on $\mathbb{R}$.*

b. *Show that* $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ *maps the imaginary axis $\mathbb{I}$ to the semicircle with centre $1/2$ joining 0 to 1.*

c. *Let $\xi < \eta \in \mathbb{R}$. Find a map $T \in PSL(2,\mathbb{R})$ which maps 0 to ξ and ∞ to η.*

d. *Why is any semicircle with centre on $\mathbb{R}$ a geodesic (straight line) in $\mathbb{H}$?*

e. *Show that there is a* unique *geodesic joining any two points in $\mathbb{H}$, namely the semicircle through the two points with centre on $\mathbb{R}$.*

The group $SL(2, \mathbb{Z})$

The group $SL(2, \mathbb{Z})$ is the subgroup of $SL(2, \mathbb{R})$ all of whose entries are integers. We define $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z}) / \pm Id$.

Exercise 1.5. a. *Show that* $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *is the unique non-trivial element of* $SL(2, \mathbb{Z})$ *which maps* $\mathbb{I}$ *to itself.*

b. *Check that, as an isometry of* $\mathbb{H}$*,* J *has order* 2 *and fixes* i*.*

The 'tiles' of $\mathcal{F}$ are all *ideal triangles*. This means that each tile has three geodesic sides, which meet in pairs on the boundary at infinity, i.e. $\mathbb{R} \cup \infty$. We denote the triangle with vertices $0, 1, \infty$ by Δ, called the *basic triangle*. When we need to be strict, we consider that Δ is the closed triangle including its sides (but excluding the 3 vertices which lie outside $\mathbb{H}$) and we let Δ° denote its interior, that is, Δ excluding its sides.

Exercise 1.6. a. *Find the element* $S \in PSL(2, \mathbb{Z})$ *which sends* $0 \to 1, 1 \to \infty, \infty \to 0$.

b. *Conclude that the stabiliser of* Δ *in* $PSL(2, \mathbb{Z})$ *has order* 3.

c. *Show that* S *has a unique fixed point in* $\mathbb{H}$*, and find it.*

The following proposition allows us to prove the key facts about $\mathcal{F}$.

Proposition 1.2. *The ideal triangles in the Farey tessellation* $\mathcal{F}$ *cover the hyperbolic plane without overlaps (except of their boundaries). Moreover if* $g \in SL(2, \mathbb{Z})$*, then* $g(\Delta)$ *is a triangle in* $\mathcal{F}$*.*

Proof. From the construction, it is clear that every point in $\mathbb{H}$ is contained in at least one (closed) ideal triangle of the construction. We have to show that no two triangles overlap.

First note that every triangle in the tessellation is the image of Δ under some element in $SL(2, \mathbb{Z})$. In fact by Exercise 1.1, if $p/q, r/s$ are joined by an arc of $\mathcal{F}$ and if we assume that $p/q > r/s$ then $\det \begin{pmatrix} p & r \\ q & s \end{pmatrix} = 1$ so that $T = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in SL(2, \mathbb{Z})$. By Exercise 1.4, T carries the positive imaginary axis $\mathbb{I}$ to the hyperbolic line joining p/q to r/s, in other words, the semicircle with these endpoints. Moreover T carries 1 to the point $p/q \oplus r/s$ so that it takes the other two sides of Δ to semicircular arcs joining these new neighbours.

Let $\mathcal{T}$ be the set of triangles in $\mathcal{F}$. If $E \in \mathcal{T}$, let E° denote its interior. We have to show that $E_1^\circ \cap E_2^\circ = \emptyset$ for any $E_1, E_2 \in \mathcal{T}$. We have just shown that $E_i = g_i(\Delta)$ for some $g_i \in SL(2, \mathbb{Z})$. So it is enough to show that $\Delta^\circ \cap g(\Delta^\circ) = \emptyset$ for any $g \in SL(2, \mathbb{Z})$. (Why?) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $a/c > b/d$. By translating and rotating Δ if needed (using the transformation S of Exercise 1.6), we may assume that the side of $g(\Delta)$ joining

a/c to b/d cuts the imaginary axis $\mathbb{I}$ (why?), so that $a/c > 0 > b/d$. We claim this is impossible for $g \in SL(2,\mathbb{Z})$. Note that without loss of generality we can take $d > 0$, (why?) so automatically $b < 0$. Then a, c have the same sign. If both are positive then $1 = ad - bc \geq 1 + 1 = 2$ which is impossible. The other case is similar.

The same argument shows that $g(\Delta) \in \mathcal{T}$ for any $g \in SL(2,\mathbb{Z})$. This completes the proof. $\square$

Here are some important consequences of Proposition 1.2.

Corollary 1.3. 1. *Every pair of neighbouring rationals are the endpoints of some side of $\mathcal{F}$.*

2. *Every point $p/q \in \mathbb{Q}$ is a vertex of $\mathcal{F}$.* Hint: Use the Euclidean algorithm!

3. *The Farey tessellation $\mathcal{F}$ is invariant under the action of $PSL(2,\mathbb{Z})$.*

Exercise 1.7. *Prove Corollary 1.3. Hint for (2): Use the Euclidean algorithm!*

Exercise 1.8. a. *Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as in Exercise 1.5. Suppose that $g \in PSL(2,\mathbb{Z})$ carries $\mathbb{I}$ to another side s of $\mathcal{F}$, so that $g(i) \in s$. Prove that gJg^{-1} is the unique non-trivial element in $PSL(2,\mathbb{Z})$ which fixes s.*

b. *Find an element $g \in SL(2,\mathbb{Z})$ which carries $\mathbb{I}$ to the hyperbolic line from 0 to 1 and hence or otherwise, find the unique non-trivial element of $PSL(2,\mathbb{Z})$ which fixes the point $(1+i)/2$.*

Exercise 1.9. a. *Explain why J maps any hyperbolic line through i to itself, interchanging endpoints.*

b. *With g as in Exercise 1.8, prove that $T = gJg^{-1}J = \pm \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ maps the hyperbolic line L joining i to $(1+i)/2$ to itself. Hint: T is the product of two π rotations about points on L.*

c. *What are the end points of this line? Check they are fixed by T.*

We will come back to this transformation T later.

Finally, here is an exercise on hyperbolic geometry which we will need in the last lecture.

Exercise 1.10. a. *Let H be the region above the horizontal line $\Im z = h$. Explain why the image of H under $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the region inside a disk tangent to $\mathbb{R}$ at a/c.*

b. *Prove that the radius of this disk is $1/2hc^2$. Hint: Suppose the disk has radius r, so its highest point is $a/c + 2ir$. Explain why h is the imaginary part of $T^{-1}(a/c + 2ir)$ and hence find the formula relating r and h.*

Mathematical billiards

by

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Lectures at the LMS Undergraduate Summer School
Loughborough, July 2015

This mine-course is an introduction to mathematical billiards. I shall start with a motivation: a mechanical system with elastic collisions is interpreted as a billiard. I plan to cover the following topics: variational approach to billiard trajectories, the area preserving property of the billiard map, geometry of the space of oriented lines, billiards in conics and geometrical consequences, including the Poncelet Porism. About 30 exercises will be offered.

Recommended literature:

V. Kozlov, D. Treshchev. Billiards. A genetic introduction to the dynamics of systems with impacts. AMS, Providence, RI, 1991.

S. Tabachnikov. Geometry and billiards. AMS, Providence, RI, 2005.

N. Chernov, R. Markarian. Chaotic billiards. AMS, Providence, RI, 2006.

LMS. Billiards: Exercises

1. Find a 6-periodic billiard trajectory in every right triangle and interpret it as a periodic motion of two mass points on a segment, subject to elastic collisions.

2. Consider the motion of three mass points m_1, m_2, m_3 on a circle, subject to elastic collisions. Assume that the center of mass of the points has zero angular speed. Prove that this is the billiard in an acute triangle; find the angles of this triangle.

3. Prove that the foot points of the altitudes of an acute triangle form therein a 3-periodic billiard trajectory (Fagnano).

4. In which angles can the billiard reflection be continuously defined for the trajectories that hit the corner?

5. Consider the elastic collision of two identical balls in $\mathbb{R}^3$, one at rest and the other moving. Show that after collision they will move in orthogonal directions.

6. Consider two identical discs of radius r on the torus $\mathbb{R}^2/\mathbb{Z}^2$, with a fixed center of mass and subject to elastic collisions. Describe the motion of this system as a 2-dimensional billiard.

7. Deduce Snell's Law from Fermat's Principle.

8. Prove that a smooth convex plane body has at least two diameters. What about dimension 3? Dimension n?

9. (i) A (plane) periscope is a system of two mirrors that send a (say) vertical beam of light to a vertical beam, inducing an invertible transformation of one beam to another (that is, a transformation of their normal sections). Given such a local transformation of segments, show that there exists a periscope that realizes it.

(ii)* Which local transformation of 2-dimensional disc can be realized by a periscope in $\mathbb{R}^3$?

10. Prove that the set of 2-periodic billiard trajectories has zero area (with respect to the area form ω).

11. What is the topology of the space of non-oriented lines in the plane? Of non-oriented great circles on the sphere?

12. How does the change of origin affect the coordinates on the space of oriented lines?

13. (i) Prove that, up to a factor, ω is the only isometry-invariant area form on the space of oriented lines.
(ii) Does the space of oriented lines in $\mathbb{R}^2$ have an isometry-invariant metric? What about the space of oriented great circles on the sphere?

14. What is the effect of refraction (subject to Snell's Law) on ω?

15. Let Γ be a closed convex plane curve, and γ a closed, possibly self-intersecting, curve inside Γ; let L and ℓ be their lengths. Prove that there exists a line that intersects γ at least $[2\ell/L]$ times.

16. The distance between the lines on a ruler paper is 1. Find the probability that a needle of length 1, randomly dropped on the paper, intersects a line. What is the expected number of intersections for a needle of length L? (Buffon's needle problem).

17. (i) Formulate and prove Crofton's formula for S^2. Apply it to prove that the total curvature of a closed space curve is not less than 2π.
(ii)* Prove that the total curvature of a knotted curve in $\mathbb{R}^3$ is not less than 4π (Fáry-Milnor theorem).

18. Find the average area of a plane projection of a unit cube.

19. Define the *length* of a rectangular box as the sum of its dimensions. Can a box of smaller length contain a box of greater length? Consider 2- and 3-dimensional versions of the problem.

20. Prove that the geometric and analytic definitions of confocal conics are equivalent.

21. Prove that a billiard trajectory in an ellipse that intersect the segment between the foci remains tangent to a confocal hyperbola.

22. Prove that the geometric and analytic formulations of integrability of the billiard inside an ellipse are equivalent.

23. Prove that the product of distances from the foci of an ellipse to a segment of the billiard trajectory is an integral of the billiard ball map.

24. Consider an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and consider the diagonal linear map that takes it to a confocal ellipse. Show that the points related by this map lie on a confocal hyperbola.

25. Let F_1 and F_2 be the foci of an ellipse. The billiard reflection gives a transformation of the lines emanating from F_1 to the lines through F_2. Identify pencils of lines with the projective line $\mathbf{RP}^1$ via the stereographic projection. Show that the resulting transformation of $\mathbf{RP}^1$ is Möbius (fractional-linear). Deduce that the billiard trajectory through the foci tends to the great axis of the ellipse.

26. Find a geometrict proof of 'the most elementary theorem of elementary geometry'.

27. Prove the Euler-Fuss relations for $n = 3$ and $n = 4$:

$$\frac{1}{R-a} + \frac{1}{R+a} = \frac{1}{r}; \quad \frac{1}{(R-a)^2} + \frac{1}{(R+a)^2} = \frac{1}{r^2},$$

where $R > r$ are the radii of the circles and a is the distance between their centers.

Groups, graphs and virology

by

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Lectures at the LMS Undergraduate Summer School
Loughborough, July 2015

This lecture course will cover the group theoretical underpinnings of virus architecture. After a brief introduction to symmetry groups and Caspar and Klugs approach to the modeling of viral capsids in terms of spherical graphs and tilings, we will introduce non-crystallographic Coxeter groups and their affine extensions. We will demonstrate how these can be used to predict the organization of material in viruses at different radial levels simultaneously, including capsid structure and genome organization. We will moreover show that these group theoretical techniques can also account for the atomic positions in nested carbon cage structures called fullerenes, thus demonstrating that mathematics developed for specific applications can have much wider impact in Science.

A background in group theory is not required, but would be an advantage. In the colloquium talk, we will demonstrate how these insights can be used to better understand how viruses form and infect their hosts, and thus underpin the development of novel anti-viral therapies.

Recommended literature:

J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge University Press, 1990.

T. Keef, J.P. Wardman, N.A. Ranson, P.G. Stockley and R. Twarock, Structural constraints on the three-dimensional geometry of simple viruses: case studies of a new predictive tool, Acta Crystallogr A. 69, 140-50, 2013.

P. Dechant, J. Wardman, T. Keef and R. Twarock, Viruses and fullerenes - symmetry as a common thread? Acta Cryst A 70:162-7, 2014.

D.L. Caspar and A. Klug, Physical Principles in the Construction of Regular Viruses, Cold Spring Harb. Symp. Quant. Biol. 27, 124, 1962.

Exercises for Mathematical Virology

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1. *Warm-up question:* Which of the following tables are symmetry tables of a group? State one axiom that fails to hold if the table does not represent a group. If it is a group, state the symmetry operations that correspond to the group elements?

(a)

	A	B	C	D
A	D	C	B	A
B	C	D	A	B
C	B	A	D	C
D	A	B	C	D

(b)

	A	B	C	D
A	A	B	C	D
B	D	A	B	C
C	C	D	A	B
D	B	C	D	A

(c)

	A	B	C	D
A	A	B	C	D
B	B	A	A	D
C	D	A	C	D
D	B	C	D	C

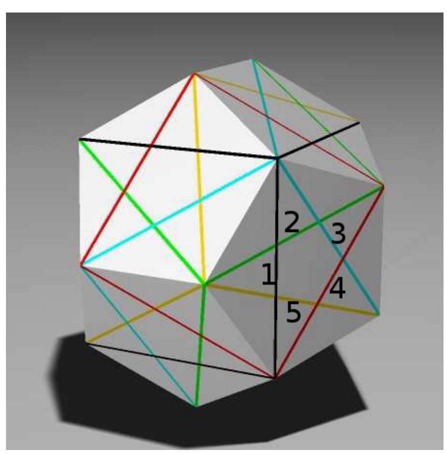
(d)

	A	B	C	D	E
A	E	C	D	B	A
B	C	D	A	E	B
C	D	E	B	A	C
D	B	A	E	C	D
E	A	B	C	D	E

2. *Icosahedral symmetry and permutation groups:*

   (a) Show that the 3-cycles in S_5 generate the subgroup A_5.

   (b) Use this result to show that the rotational symmetry group of the icosahedron is isomorphic to A_5.
   In order to do this, consider the following picture:

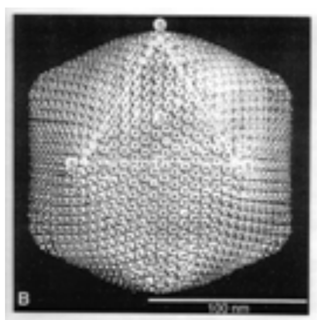


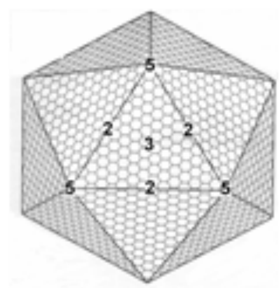
   and work through the steps listed on the next page.

i. Explain what is meant by duality of the dodecahedron and the icosahedron. What does this imply for the symmetry groups of these Platonic solids? In particular, what is the order of the symmetry group of the dodecahedron?
ii. What is the order of A_5?
iii. Consider the 5 cubes inscribed into a dodecahedron as in the figure. Number the cubes by 1 to 5 as marked: black: 1; green: 2; blue: 3; red: 4; yellow: 5. Use this to explain why each rotation of the dodecahedron corresponds to an element of S_5.
iv. By considering rotations about axes which join opposite pairs of vertices, show that every 3-cycle in S_5 is generated in this way.
v. Use the result of part (a) to argue that the rotational symmetry group of the icosahedron is isomorphic to A_5.

Note: The figure has been adapted from:
http://www.chiark.greenend.org.uk/∼sgtatham/polypics/dodec-cubes.html

3. *Generators of icosahedral symmetry:* Viruses share their rotational symmetries with the icosahedron.



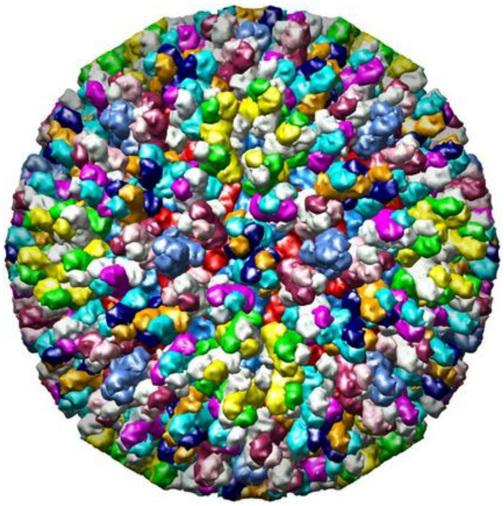


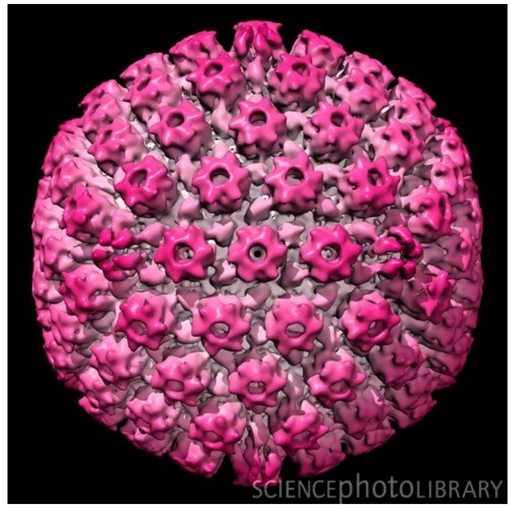
The choice of generators of the icosahedral group is not unique. In the lecture we have discussed the generators g_2, g_3 with $(g_2 g_3)^5 = 1$. Consider here the following options:

(a) g_2, g_5 with $(g_2 g_5)^3 = 1\,,$
(b) g_3, g_5 with $(g_3 g_5)^2 = 1\,.$

For each, indicate explicitly the group elements corresponding to the copies of the fundamental domain tesselating the five triangular faces of the icosahedron around the 5-fold axis representing g_5 (i.e. three group elements per triangular face, and hence 15 group elements in total). As in the lecture, indicate them explicitly on a drawing that represents those 5 faces, and mark also the location of the symmetry axis corresponding to the second generator.

4. *Viruses and symmetry 1:* Bluetongue (left) is a T = 13 virus and Herpes Simplex (right) is a T = 16 virus.





For each of these viruses:

(a) determine the number of proteins and protein clusters of different types in the capsid

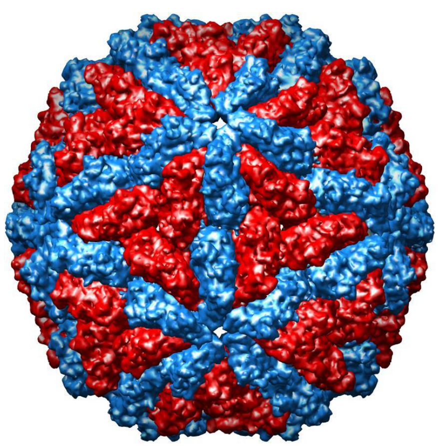
(b) determine whether their protein organisation has handedness

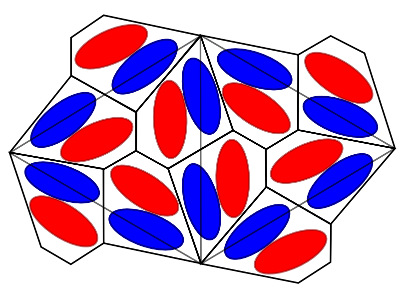
(c) draw their surface lattices, i.e the hexagonal Caspar-Klug lattice superimposed on one of the 20 triangular faces of the icosahedron and mark the locations of the icosahedral symmetry axes.

Are there any other T-numbers between $T = 13$ and $T = 16$?

5. *Viruses and symmetry 2:* Bacteriophage HK97 is a $T = 7l$ virus obeying a rhomb tiling. What is the number of proteins, pentamers and hexamers in its viral capsid?

6. *Viruses and symmetry 3:* L-A Virus is a virus infecting yeast:





Given the tiling above, which is shown superimposed on two of the icosahedral triangular faces, answer the following questions:

(a) How many proteins are there in the viral capsid?

(b) Does this correspond to a Caspar-Klug triangulation number?

(c) Does this virus have handedness?

7. *Non-crystallographic Coxeter groups and root systems:* The root system of the non-crystallographic Coxeter group H_2 is given by

$$\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \tau\alpha_2), \pm(\alpha_2 + \tau\alpha_1), \pm(\tau\alpha_1 + \tau\alpha_2)\}$$

where $\tau := \frac{1}{2}(1+\sqrt{5}) = 2\cos\frac{\pi}{5}$, with simple system $\Delta = \{\alpha_1, \alpha_2\}$. Note that Φ consists of the vectors pointing to the vertices of a decagon.

(a) Using the definition of a Cartan matrix in terms of scalar products between the vectors in the simple system (simple roots), convince yourself that the off-diagonal entries in the Cartan matrix of H_2 are given by $-\tau$.

(b) Via an affine extension an extra row ($a_{0j},\ j = 0, 1, 2$) and column ($a_{j0},\ j = 0, 1, 2$) are introduced in the Cartan matrix. Let $\alpha_0 := -\tau(\alpha_1 + \alpha_2)$ and consider the Cartan matrix for $\{\alpha_0, \alpha_1, \alpha_2\}$. Show that the off-diagonal elements a_{01} and a_{02} of this matrix are both equal to $1 - \tau$.

8. *Affine extensions of Coxeter groups:* An affine extended Coxeter group contains off-centre reflections (affine reflections) that act on a vector v as follows:

$$r_\alpha^{aff} v = v + \alpha - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha\,.$$

Let $r_\alpha v = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha$ denote a reflection at a central plane orthogonal to α. Show that composition of these two operations acts as a translation by α, i.e. show that

$$T_\alpha v := r_\alpha^{aff} r_\alpha v = v + \alpha\,.$$