LMS Undergraduate Summer School 2016

The many faces of polyhedra

1. Pick's theorem

Let $\mathcal{L} = \mathbb{Z}^2 \subset \mathbb{R}^2$ be integer lattice, P be polygon with vertices in $\mathcal{L}$ (**integral polygon**)

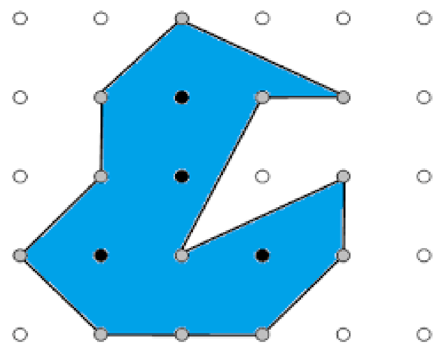


Figure 1. An example of integral polygon

Let I and B be the number of lattice points in the interior of P and on its boundary respectively. In the example shown above $I = 4$, $B = 12$.

Pick's theorem. *For any integral polygon P its area A can be given by Pick's formula*

$$A = I + \frac{B}{2} - 1. \tag{1}$$

In particular, in our example $A = 4 + \frac{12}{2} - 1 = 9$, which can be checked directly.

The following example (due to Reeve) shows that no such formula can be found for polyhedra. Consider the tetrahedron T_h with vertices $(0,0,0), (1,0,0), (0,1,0), (1,1,h)$, $h \in \mathbb{Z}$ (**Reeve's tetrahedron**, see Fig. 2).

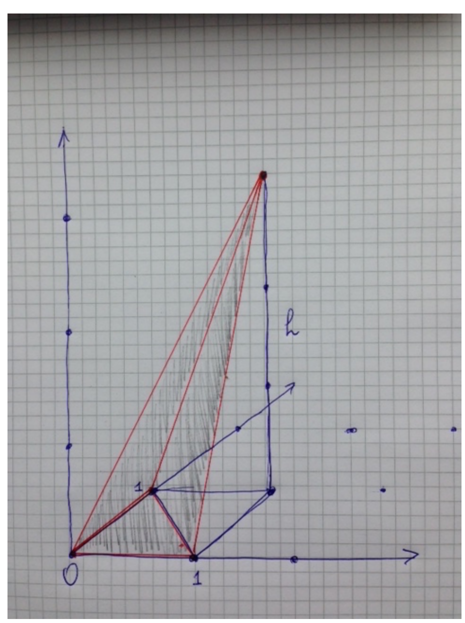


Figure 2. Reeve's tetrahedron T_h

It is easy to see that T_h has no interior lattice points and 4 lattice points on the boundary, but its volume is $\mathrm{Vol}(T_h) = h/6$.

2. Ehrhart theory

Let $P \subset \mathbb{R}^d$ be an integral **convex polytope**, which can be defined as the convex hull of its vertices $v_1, \dots, v_N \in \mathbb{Z}^d$:

$$P = \{x_1 v_1 + \dots x_N v_N, x_1 + \dots x_N = 1, x_i \geq 0\}.$$

For $d = 2$ and $d = 3$ we have convex polygon and convex polyhedron respectively.
Define

$$L_P(t) := |tP \cap \mathbb{Z}^d|,$$

which is the number of lattice points in the scaled polytope tP, $t \in \mathbb{Z}$.

Ehrhart theorem. $L_P(t)$ *is a polynomial in* t *of degree* d *with rational coefficients and with highest coefficient being volume of* P:

$$L_P(t) = \mathrm{Vol}(P) t^d + \dots + 1.$$

$L_P(t)$ is called **Ehrhart polynomial**.
Define also **Ehrhart series** by

$$\mathrm{Ehr}_P(z) = \sum_{t \in \mathbb{Z}_{\geq 0}} L_P(t) z^t.$$

Example 1. Let $\square_d$ be the d-dimensional unit cube:

$$\square_d = \{(x_1, \dots, x_d) : 0 \leq x_i \leq 1\} = [0,1]^d.$$

Then $t\square_d = [0,t]^d$ and Ehrhart polynomial is

$$L_{\square_d}(t) = (t+1)^d.$$

The Ehrhart series is

$$\mathrm{Ehr}_{\square_d}(z) = \sum_{t \geq 0} (t+1)^d z^t = \frac{1}{z} \sum_{j \geq 1} j^d z^j = \frac{1}{z} \left(z \frac{d}{dz} \right)^d \frac{1}{1-z}.$$

In particular,

$$\mathrm{Ehr}_{\square_1}(z) = \frac{d}{dz} \frac{1}{1-z} = \frac{1}{(1-z)^2},$$

$$\mathrm{Ehr}_{\square_2}(z) = \frac{1}{z} \left(z \frac{d}{dz} \right)^2 \frac{1}{1-z} = \frac{z+1}{(1-z)^3}$$

$$\mathrm{Ehr}_{\square_3}(z) = \frac{1}{z} \left(z \frac{d}{dz} \right)^3 \frac{1}{1-z} = \frac{z^2+4z+1}{(1-z)^4}$$

$$\mathrm{Ehr}_{\square_4}(z) = \frac{1}{z} \left(z \frac{d}{dz} \right)^4 \frac{1}{1-z} = \frac{z^3+11z^2+11z+1}{(1-z)^5}$$

The coefficients of the numerators are known as **Eulerian numbers** $A(d,k)$, which count the permutations of $\{1, 2, \dots, d\}$ with k ascents.

Example 2. Let Δ_d be the standard d-dimensional *simplex*:

$$\Delta_d = \{(x_1, \dots, x_d) : x_1 + \dots + x_d \leq 1,\ x_i \geq 0\}.$$

Then

$$t\Delta_d = \{(x_1, \dots, x_d) : x_1 + \dots + x_d \leq t,\ x_i \geq 0\}.$$

The Ehrhart polynomial is

$$L_{\Delta_d}(t) = \binom{d+t}{d} = \frac{(t+d)(t+d-1)\dots(t+1)}{d!}$$

and the Ehrhart series is

$$\mathrm{Ehr}_{\Delta_d}(z) = \sum_{t \geq 0} \binom{d+t}{d} z^t = \frac{1}{(1-z)^d}.$$

Example 3. For integral polygon P

$$L_P(t) = At^2 + \frac{B}{2} t + 1,$$

where A is the area and B the number of boundary points of P.
The Ehrhart series is

$$\mathrm{Ehr}_P(z) = \frac{(A - \frac{B}{2} + 1) z^2 + (A + \frac{B}{2} - 2) z + 1}{(1-z)^3}.$$

3. Ehrhart-Macdonald reciprocity

Let P be an integral convex polytope and define **interior Ehrhart polynomial** $L_{P^o}(t)$ as the number of interior lattice points in tP.

There is a remarkable **Ehrhart-Macdonald reciprocity**: for any convex integral polytope of dimension d

(2) $$\boxed{L_{P^o}(t) = (-1)^d L_P(-t),}$$

(3) $$\boxed{\mathrm{Ehr}_{P^o}(z) = (-1)^{d+1}\mathrm{Ehr}_P(\frac{1}{z})}$$

where the latter is understood as equality of rational functions (but not formal series).

Example. For unit cube we have

$$L_{\square_d^o}(t) = (t-1)^d = (-1)^d(-t+1)^d = (-1)^d L_{\square_d}(-t).$$

For a polygon P

$$L_{P^o}(t) = At^2 - \frac{B}{2}t + 1 = L_P(-t).$$

4. Exercises-I

1. Prove Pick's formula for any integral triangle. Hint: see Fig.3.

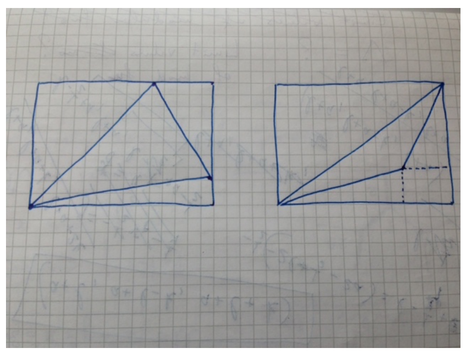


Figure 3. Embedding of an integral triangle into a rectangle

2. Find Ehrhart polynomial and Ehrhart series for the standard octahedron O with the vertices $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$ and pyramid P (see Figure 4).

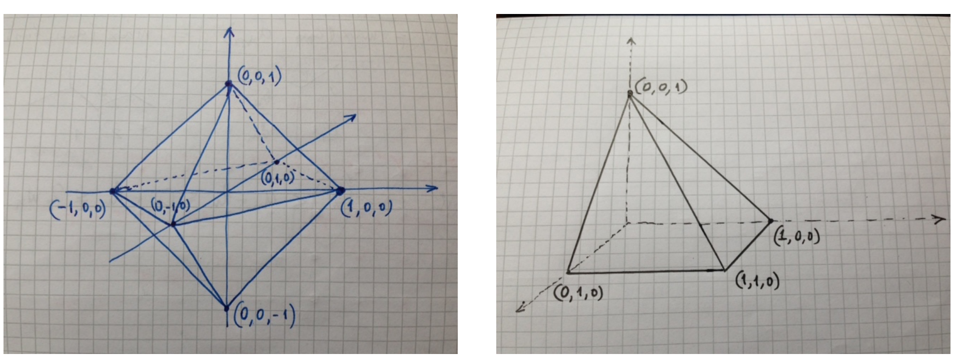


Figure 4. Standard octahedron O and pyramid P

3. Check the Ehrhart-Macdonald reciprocity for the standard d-simplex Δ_d.

4. Find the Ehrhart polynomials $L_{T_h}(t)$ for Reeve's tetrahedron and show that its coefficients could be negative. Use the Ehrhart-Macdonald reciprocity to find $L_{T_h^o}(t)$.

5. Generating functions of sets

Let $S \subset \mathbb{R}^d$ be a subset. The **generating function** (or **integer-point transform**) of S is defined by

$$\sigma_S(z) := \sum_{m \in S \cap \mathbb{Z}^d} z^m,$$

where $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$, $z = (z_1, \dots, z_d) \in \mathbb{C}^d$ and

$$z^m = z_1^{m_1} \dots z_d^{m_d}.$$

Example. For $d = 1$ and $S = [0, +\infty)$ we have

$$\sigma_S(z) = \sum_{m \in \mathbb{Z}_{\geq 0}} z^m = \frac{1}{1-z},$$

for $S = [a, b],\ a, b \in \mathbb{Z}$ we have

$$\sigma_S(z) = \sum_{a \leq m \leq b} z^m = \frac{z^a - z^{b+1}}{1-z}.$$

A **cone** $K \subset \mathbb{R}^d$ is a set of the form

$$K = \{v + \lambda_1 w_1 + \dots + \lambda_N w_N : \lambda_i \geq 0\},$$

v is called **apex** and w_k are **generators** of the cone K. The cone is called **rational** if $w_k \in \mathbb{Z}^d$ for all $k = 1, \dots, N$. We will consider only d-dimensional cones (or, d-cones) with $w_1, \dots, w_N$ spanning the whole $\mathbb{R}^d$. The d-cone K is **simplicial** if $N = d$.

For simplicial cones the generating functions can be computed effectively. We demonstrate this in 2 dimensions. Let $v = (0,0), w_1 = (1,1), w_2 = (-2,3)$, see the corresponding cone K on Fig.5. The half-open set

$$\Pi := \{\lambda_1 w_1 + \lambda_2 w_2 : 0 \leq \lambda_1, \lambda_2 < 1\}$$

is called the **fundamental parallelogram**.

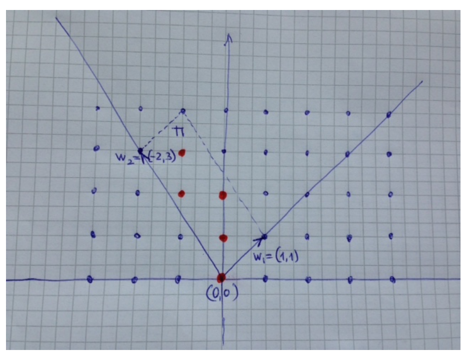


Figure 5. The cone K with its fundamental parallelogram Π

The generating function of the lattice generated by w_1 and w_2 has the form

$$\sigma(z) = \sum_{m \in \mathbb{Z}^2_{\geq 0}} z^{m_1 w_1 + m_2 w_2} = \frac{1}{(1 - z^{w_1})(1 - z^{w_2})} = \frac{1}{(1 - z_1 z_2)(1 - z_1^{-2} z_2^3)}.$$

Adding the contribution from the lattices shifted by the integer points inside Π we have the full generating function

$$\sigma_K(z) = \frac{1 + z_2 + z_2^2 + z_1^{-1} z_2^2 + z_1^{-1} z_2^3}{(1 - z_1 z_2)(1 - z_1^{-2} z_2^3)}.$$

Note that the numerator is simply the generating function of Π:

$$\sigma_\Pi(z) = 1 + z_2 + z_2^2 + z_1^{-1} z_2^2 + z_1^{-1} z_2^3.$$

For a general rational simplicial d-cone $K = \{\lambda_1 w_1 + \dots + \lambda_d w_d : \lambda_i \geq 0\}$, we have

$$\boxed{\sigma_{v+K} = \frac{\sigma_{v+\Pi}(z)}{(1 - z^{w_1}) \dots (1 - z^{w_d})},} \tag{4}$$

where Π is **half-open fundamental parallelepiped**

$$\Pi := \{\lambda_1 w_1 + \dots + \lambda_d w_d : 0 \leq \lambda_1, \dots, \lambda_d < 1\}.$$

6. Cone of P and proof of Ehrhart's theorem

Let us embed polytope $P \subset \mathbb{R}^d$ with vertices $v_1, \dots, v_N \in \mathbb{R}^d$ into $\mathbb{R}^{d+1}$ by adding $x_{d+1} = 1$:

$$w_1 = (v_1, 1), \dots, w_N = (v_N, 1)$$

and consider the cone

$$\mathrm{cone}(P) = \{\lambda_1 w_1 + \dots + \lambda_d w_d : \lambda_i \geq 0\}.$$

The original P can be recovered from the cone of P by cutting it with the hyperplane $x_{d+1} = 1$, while its scaled versions tP are intersections with $x_{d+1} = t$ (see Figure 5).

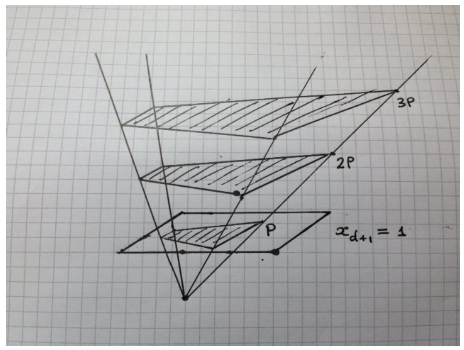


Figure 6. The cone of P with the dilates of P

Now the key observation is that

$$\sigma_{\mathrm{cone}(P)}(1, \dots, 1, z_{d+1}) = 1 + \sum_{t \geq 1} \sigma_{tP}(1, \dots, 1) z_{d+1}^t = 1 + \sum_{t \geq 1} |tP \cap \mathbb{Z}^d| z_{d+1}^t = 1 + \sum_{t \geq 1} L_P(t) z_{d+1}^t,$$

so

$$\boxed{\sigma_{\mathrm{cone}(P)}(1, \dots, 1, z) = 1 + \sum_{t \geq 1} L_P(t) z^t = \mathrm{Ehr}_P(z).} \tag{5}$$

Now we can prove Ehrhart's theorem as follows.

First triangulate P into simplices to reduces the claim to the case when P is a simplex. From formula (4) we can deduce that

$$\sigma_{\mathrm{cone}(P)}(1, \dots, 1, z) = \frac{\sigma_\Pi(1, \dots, 1, z)}{(1 - z)^{d+1}}$$

with $\sigma_\Pi(1, \dots, 1, z)$ being polynomial in z of degree at most d. Now the theorem follows from the claim that the Taylor coefficients a_k of a rational function

$$\frac{p(z)}{(1 - z)^{d+1}} = \sum_{k \geq 0} a_k z^k$$

with $p(z)$ polynomial of degree d such that $p(1) \neq 0$, are polynomial in k of degree d.

From the theory of Riemann integral we have

$$\mathrm{vol}(P) = \lim_{N \to \infty} \frac{|P \cap (\frac{1}{N}\mathbb{Z})^d|}{N^d} = \lim_{N \to \infty} \frac{|NP \cap \mathbb{Z}^d|}{N^d} = \lim_{N \to \infty} \frac{L_P(N)}{N^d} = c_d,$$

where

$$L_P(k) = c_d k^d + \dots c_0.$$

Note that we know also $c_0 = 1$, but other coefficients of Ehrhart polynomial are a bit of mystery, and can be even negative as we have seen in Reeve's case.

It is interesting that for the numerator of the Ehrhart series all the coefficients are non-negative:

Stanley's non-negativity theorem. *For an integral convex polytope* P *with*

$$\mathrm{Ehr}_P(z) = \frac{h_d z^d + h_{d-1} z^{d-1} + \dots + h_0}{(1 - z)^{d+1}}$$

we have $h_0, h_1, \dots, h_d \geq 0$.

For a simplex P this follows from the interpretation of h_k as the number of integer points in the fundamental parallelepiped Π with $x_{d+1} = k$ (see formula (4)).

7. Tangent cones and Brion's theorem

For any vertex v of a convex polytope P one can naturally define **tangent cone** as

$$K_v := \{v + \lambda(x - v) : x \in P, \lambda \in \mathbb{R}_{\geq 0}\}$$

(see Fig. 7).

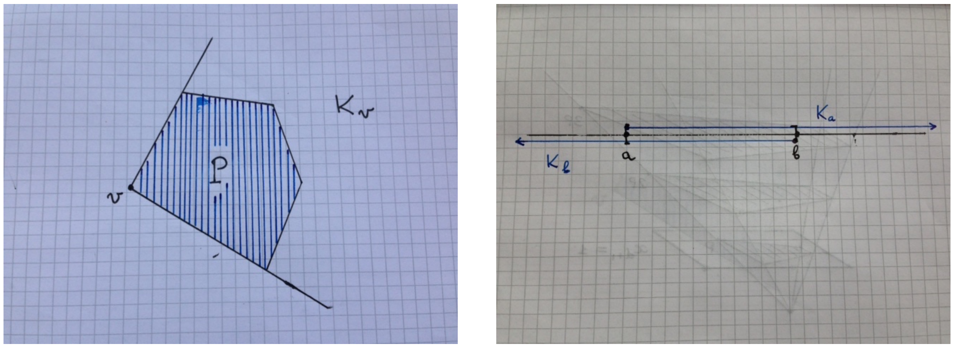


Figure 7. Tangent cones in dimension 2 and 1

Brion's theorem. *For any integral convex polytope* P *we have the identity of rational functions*

$$\boxed{\sigma_P(z) = \sum_{v \text{ a vertex of } P} \sigma_{K_v}(z)} \tag{6}$$

Example. For $P = [a, b]$ we have

$$\sigma_S(z) = \sum_{a \leq m \leq b} z^m = \frac{z^a - z^{b+1}}{1 - z} = \frac{z^a}{1 - z} + \frac{z^b}{1 - 1/z} = \sum_{m \geq a} z^m + \sum_{m \leq b} z^m = \sigma_{K_a}(z) + \sigma_{K_b}(z).$$

The general proof is based on a curious identity

$$\sum_{m \in \mathbb{Z}} z^m \equiv 0,$$

where the left hand side is understood as the sum of two rational functions:

$$\sum_{m \in \mathbb{Z}} z^m = \sum_{m \geq a} z^m + \sum_{m < a} z^m = \frac{z^a}{1 - z} + \frac{z^{a-1}}{1 - 1/z} = \frac{z^a}{1 - z} + \frac{z^a}{z - 1} \equiv 0.$$

Note that this identity does not make much sense in analysis since two last series **never converge simultaneously**!

As a corollary we have one more proof that $L_P(t)$ is polynomial. Indeed, it is enough to prove it for simplices Δ, for which we have

$$L_\Delta(t) = \lim_{z \to \mathbf{1}} \sigma_{t\Delta}(z) = \lim_{z \to \mathbf{1}} \sum_{v \text{ a vertex of } P} \sigma_{tK_v}(z) = \lim_{z \to \mathbf{1}} \sum_{v \text{ a vertex of } P} z^t v \sigma_{K_v^0}(z),$$

where K_v^0 is the cone K_v shifted to 0. We know that $\sigma_{K_v^0}(z)$ is a rational function with the denominator vanishing at $z = \mathbf{1}$ (see formula (4)), so the limit $z \to \mathbf{1}$ should be computed using the L'Hôpital's rule. It is clear that the result will polynomial in t.

8. Simple polytopes and Dehn-Sommerville relations

A convex polytope $P \subset \mathbb{R}^d$ is called **simple** if tangent cone of every vertex is simplicial.

For example, tetrahedron, cube and dodecahedron are simple, but octahedron and icosahedron are not.

Define f-**vector** as $f = (f_0, \dots, f_d)$, where f_k is the number of k-dimensional faces with $f_d := 1$. For example, f-vector of cube is $(8, 12, 6, 1)$.

Can one describe all possible f-vectors of convex polytopes? In turns out that for simple polytopes it is possible.

Define f-**polynomial** as

$$f(t) := \sum_{j=0}^{d} f_j t^j$$

and h-**polynomial** as

$$h(t) := f(t - 1) = \sum_{j=0}^{d} h_j t^j.$$

The coefficients h_k satisfy remarkable **Dehn-Sommerville relations**

(7) $$\boxed{h_k = h_{d-k}.}$$

In particular, $h_0 = \sum_{j=0}^{d}(-1)^j f_j = h_d = 1$, which implies the d-dimensional **Euler relation**

$$\chi := \sum_{j=0}^{d-1}(-1)^j f_j = 1 + (-1)^{d-1}.$$

The right hand side is called **Euler characteristic** of the boundary ∂P, which is topologically equivalent to a sphere S^{d-1}. The claim is that it does not depend on P (in that case P need not to be simple).

The classical Euler relation corresponds to the polyhedral case $d = 3$:

$$f_0 - f_1 + f_2 = 2.$$

It is known (due to Stanley) that h_k must satisfy the inequalities

(8) $$\boxed{h_0 \le h_1 \le \cdots \le h_{[d/2]},}$$

but there are additional (McMullen's) inequalities to be satisfied to make these conditions necessary and sufficient for vector f to be f-vector of a simple convex polytope.

Example. For the cube we have

$$f(t) = t^3 + 6t^2 + 12t + 8,$$
$$h(t) = (t-1)^3 + 6(t-1)^2 + 12(t-1) + 8 = t^3 + 3t^2 + 3t + 1.$$

9. Exercises-II

1. Find the generating function $\sigma_{v+K}(z_1, z_2)$ of the cone $v + K$ with $v = (-3, 1)$ and

$$K = \{\lambda_1(2,1) + \lambda_2(-1,3) : \lambda_1, \lambda_2 \ge 0\}.$$

2. For a rational cone $K = \{\lambda_1 w_1 + \lambda_2 w_2 : \lambda_1, \lambda_2 \ge 0\} \subset \mathbb{R}^2$ consider $v \in \mathbb{R}^2$ such that the cone $v + K$ does not have any integer points on the boundary. Prove that for the fundamental parallelogram Π

$$v + \Pi = -(-v + \Pi) + w_1 + w_2,$$

and hence

$$\sigma_{-v+K}(z_1, z_2) = \sigma_{v+K}(\frac{1}{z_1}, \frac{1}{z_2}),$$

where both sides should be interpreted as rational functions. Deduce **Stanley reciprocity**

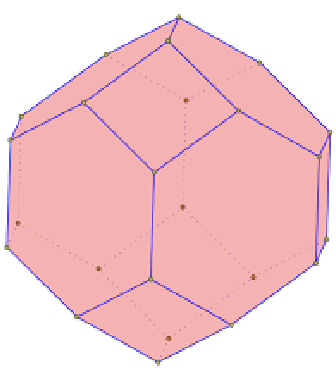
$$\sigma_{K^o}(z_1, \dots, z_d) = (-1)^d \sigma_K(\frac{1}{z_1}, \dots, \frac{1}{z_d})$$

for $d = 2$. Use Stanley reciprocity to prove Ehrhart-Macdonald reciprocity for integral convex d-polytopes

$$Ehr_{P^o}(z) = (-1)^{d+1} Ehr_P(\frac{1}{z}).$$

3. Check Brion's theorem for the triangle Δ with vertices $(0,0), (3,0), (1,3)$.

4. Compute h-polynomial for all remaining platonic solids: tetrahedron, octahedron, dodecahedron and icosahedron, and check if Dehn-Sommerville relations are satisfied. The same question for the **permutahedron** *(which is truncated octahedron) and* **truncated icosahedron** *(which is a polyhedral model of football), see Fig. 8.*



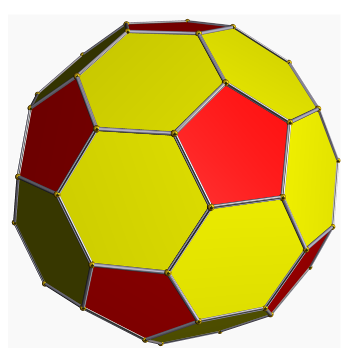


FIGURE 8. Permutahedron and truncated icosahedron