From Platonic solids to quivers

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Abstract

This course will be a whirlwind tour through representation theory, a major branch of modern algebra. We being by considering the symmetry groups of the Platonic solids, which leads naturally to the notion of a reflection group and its associated root system. The classification of these reflection groups gives us our first examples of quivers (= direct graphs). Though easy to define, we'll see that the representation theory associated to quivers is very rich. We will use quivers to illustrate the key concepts, ideas and problems that appear throughout representation theory. Coming full circle, the course will culminate with the beautiful theorem by Gabriel, classifying the quivers of finite type in terms of the root systems of reflection groups. The ultimate goal of the course is to give students a glimpse of the beauty and unity of this field of research, which is today very active in the U.K.

Contents

Introduction

This series of four lectures is designed to be a crash course in representation theory; the only background knowledge assumed is basic group theory and linear algebra. We start with the ancient Greeks and their study of the Platonic solids and end up finally with Gabriel's theorem classifying quivers with finitely many indecomposable representations. At first glance, the topics covered - the Platonic solids, finite reflection groups and quiver, seem completely unrelated. But one of the key messages of the lectures is that there is in fact a deep connection between these objects. Underlying each of their classifications is a certain graph called a Coxeter/Euler graph, and the corresponding positive definite symmetric form. Remarkably, these graphs appear in several other areas of mathematics; in section 4.6 I have listed all those that I know, but I'm certain that there are several more.

As the name suggests, the Platonic solids were well-known to the ancient Greeks. There seems to some contention as to who first discovered them (some attribute this to Pythagoras, others to Theaetetus) but they were popularized by Plato, who associated to each of the first four solids one of the four elements, "fundamental building blocks" of the universe,

$$\begin{aligned} \text{tetrahedron} &\leftrightarrow \text{fire} \\ \text{cube} &\leftrightarrow \text{earth} \\ \text{octahedron} &\leftrightarrow \text{air} \\ \text{icosahedron} &\leftrightarrow \text{water} \end{aligned}$$

The dodecahedron was meant to represent the whole universe. Later the German astronomer Johannes Kepler based his model of the solar system on the Platonic solids. For a given solid P there is a unique sphere $P \subset S$ such that all the vertices of P lie on S. There is a second, unique, sphere $S' \subset P$ such that the midpoint of each face of P lies in S'. By nesting the Platonic solids $P_1 \subset P_2 \subset \cdots \subset P_5$ in the correct way, we get six concentric spheres $S_1 \subset \cdots \subset S_6$. Each of these six spheres was meant to correspond to the orbit of the six planets that were known at that time, so that each solid separates a pair of planets,

$$\begin{aligned} \text{octahedron} &\leftrightarrow \text{Mercury and Venus} \\ \text{icosahedron} &\leftrightarrow \text{Venus and Earth} \\ \text{dodecahedron} &\leftrightarrow \text{Earth and Mars} \\ \text{tetrahedron} &\leftrightarrow \text{Mars and Jupiter} \\ \text{cube} &\leftrightarrow \text{Jupiter and Saturn;} \end{aligned}$$

see Figure 1.1. Though we know today that this model can't possibly have any real meaning, it is remarkable how close the fit actually was. In the first lecture we will focus on the symmetries of the Platonic solids. Due to the mysterious duality that the solids posses, these symmetry groups are relatively easy to calculate. Remarkably, these groups are all reflection groups i.e. they are generated by reflections.

In the second lecture, we study in more detail this notion of a reflection group. It is a classical result that one can classify all finite reflection groups. This classification result will be at the heart of the lecture. In order to state it and outline a proof, we will be forced to introduce Coxeter graphs and their corresponding symmetric bilinear forms. The fact that one can classify finite reflection groups relies on the fact that one can classify those Coxeter graphs that are positive definite. To go between reflection groups and Coxeter graphs, one needs the notion of a root system. This is a very natural object that one can associate to a reflection group, but it still contains all the information needed to reconstruct the group. For a given root system, it is easy to work out what the corresponding Coxeter graph is.

In the third lecture we leave behind Platonic solids and reflection groups and turn instead to quivers. A quiver is simply a directed graph. These were first introduced by Gabriel to encode in a very simple, combinatorial way, very difficult problems in linear algebra. This gives rise to the notion of a representation of a quiver. Though the definition is very simple, representations can turn out to have very complex structures. As such they are still the focus of much research by mathematicians today. Quivers are also the ideal introduction to representation theory - this major branch of algebra aims to understand objects, whether rings, groups, Lie algebras,.... by understanding how they act on vector spaces. Due to their combinatorial natural, Quiver representations provide a large source of interesting examples of representations and are useful in testing conjectures in the theory. They also provide us with the ideal setting in which to introduce the basic concepts of the subject such as a simple or indecomposable representation, and the notion of semi-simplicity.

In the final lecture we will focus on one of the fundamental results in the theory of quiver representations, that of Gabriel's Theorem. It's easy to see that in order to classify all the representations of a quiver it suffices to classify those that are indecomposable i.e. those that we cannot decompose into smaller representations. Therefore it is natural to ask for which quivers might we expect there to be only finitely many indecomposable representations. Gabriel's Theorem provides a complete answer to this question - it is precisely those quivers whose underlying graph is a positive definite Euler graph. This provides us (amazingly!) with a link to the second lecture on reflections groups, since the positive definite Euler graphs are exactly the positive definite Coxeter graphs with at most one edge between any two vertices. Not only that, Gabriel showed that there is a bijection between the indecomposables of the quiver and the roots in the corresponding root system. Thus, a concept introduced in the theory of reflection groups appears mysteriously in answering one of the basic questions in quiver representations. In fact, it turns out that root systems play an important role throughout the representations theory of quivers. This is exemplified by Kac's Theorem, which is a vast generalization of Gabriel's Theorem.

At the end of each section I have listed several sources where you can learn more about the topic under consideration. I hope the lecture notes will persuade you to dip into some of these. The reward for those of you who do so is the revelation of a beautiful, rich world which touches on many of the cornerstones of mathematics, such as algebraic geometry, topology, group theory and, of course, representation theory. Here I would just like to mention the survey [19] by Slodowy and the thesis [21] by van Hoboken, which are both excellent expositions of the connection between Platonic solids, finite subgroups of $SU(2)$ and Lie algebras of simply laced type.

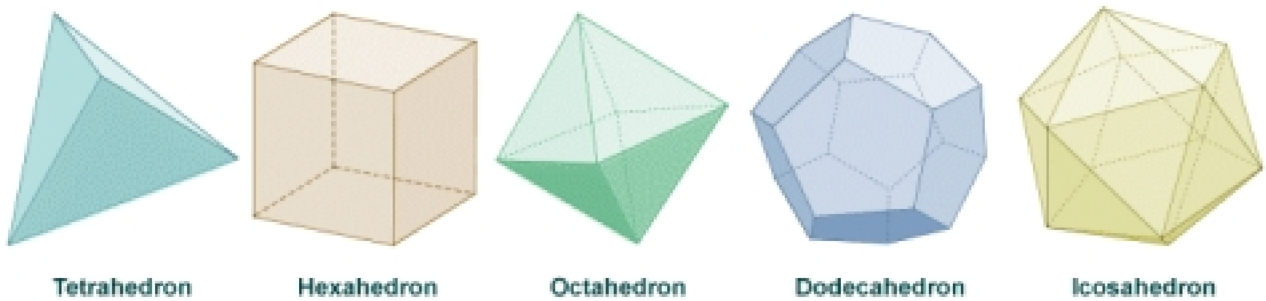


Figure 1: The five Platonic solids.

1 The Platonic solids

In the first lecture, we travel all the way back to ancient Greece, and explore the Platonic solids that so fascinated the Greeks at that time (so much so that Plato saw them as the very building blocks of the universe).

1.1 The five regular solids

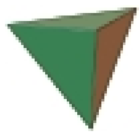
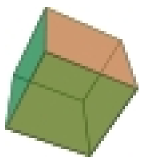
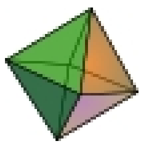
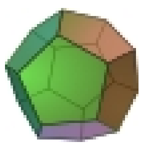
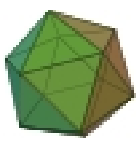
A convex regular polyhedron in $\mathbb{R}^3$ is called a *Platonic solid* or *regular solid*. Here, a *polyhedron* is a region bounded by planes in $\mathbb{R}^3$. It has two-dimensional *faces* which meet in one-dimensional *edges*, which meet in *vertices*. We shall assume that a polyhedron is bounded. This implies that it is compact as a topological space. We say that a polyhedron is *regular* if all its faces, edges and vertices are equal. That is, all the faces meet at the same angle and that the same number of edges meet at the same angles at each vertex. In particular, this implies that all the faces are the same regular polygon.

As the name suggests, the Platonic solids were popularized by Plato, therefore they are also know as the Platonic solids. In Figure 1.1, you'll see some Platonic solids. The *Schläfli symbol* symbol of a Platonic solid is $\{p, q\}$, where p is the number of faces meeting at a given vertex, each of which is a q-gon. A moment's thought should convince you that there is at most one Platonic solid with Schläfli symbol $\{p, q\}$.

Theorem 1.1. *The five polygons of Figure 1.1 are the only regular polygons.*

Proof. Let $\{p, q\}$ be the Schläfli symbol of the polygon. Then we have p faces (each a regular q-gon) meeting at every vertex. The angle at the corner of the q-gon is $(q-2)\pi/q$ and since the polyhedron is convex we must have $p \times (q-2)\pi/q < 2\pi$, which implies that $(p-2)(q-2) < 4$. The only positive integers satisfying this are $(q, p) = (3, 3), (4, 3), (3, 4), (5, 3), (3, 5)$, corresponding to the above five. $\square$

Here's some more information about them:

Polyhedron	Vertices V	Edges E	Faces F	Schläfli symbol
Tetrahedron T 	4	6	4	$\{3, 3\}$
Hexahedron H 	8	12	6	$\{4, 3\}$
Octahedron O 	6	12	8	$\{3, 4\}$
Dodecahedron D 	20	30	12	$\{5, 3\}$
Icosahedron I 	12	30	20	$\{3, 5\}$

If you're curious to know, "hedron" is derived from the Greek for face. From now on, we'll call the Hexahedron the "cube" for simplicity.

1.2 Topology

Notice that the numbers (V, E, F) satisfy a remarkable property, namely we always have

$$V - E + F = 2.$$

Euler showed that this equation actually holds for any convex polyhedron in $\mathbb{R}^3$. This is because $V - E + F$ is a *topological* property of the shapes. Topologically, any convex polyhedron is homeomorphic to the 3-sphere. Thus, the quantity $V - E + F$ will always equal 2, which is the *Euler characteristic* of the 3-sphere. More generally, for any bounded, closed polyhedron P in $\mathbb{R}^3$,

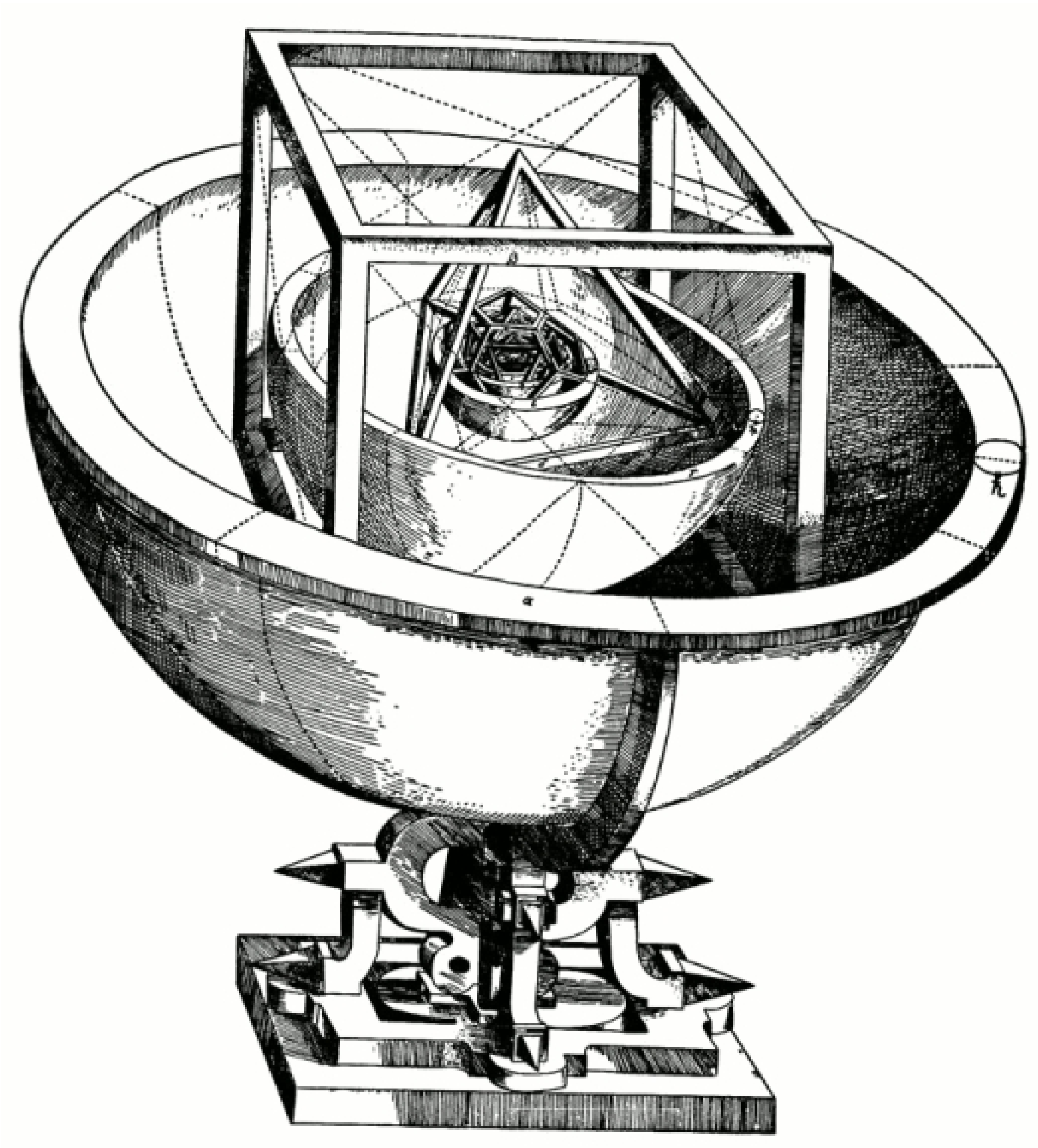


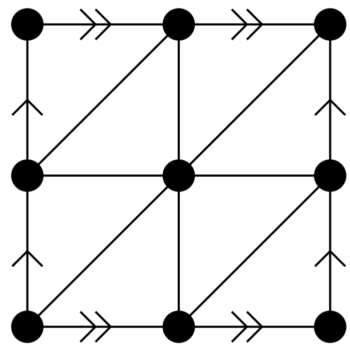
Figure 2: Kepler's model of the solar system, based on the five Platonic solids.

we have

$$V - E + F = 2 - 2g, \tag{1}$$

where g is the *genus* of P; this is the number of “holes” in P.

Exercise 1.2. Check that equation (1) holds for the Platonic solids and for the torus with triangulation



where one should glue together the two edges labeled by a single arrow and similarly for the two arrows labeled by a double arrow.

1.3 Duality

The Platonic solids possess a beautiful duality - take a solid P and construct a new polyhedron Q within P as follows. For each face of P put a vertex at the centre of this face. These will be the vertices of Q. If two faces of P share a common edge then draw a new edge from the vertex at the centre of one face to the vertex at the centre of the other face. These will be the edges of Q. Notice that there are bijections

$$\{ \text{ faces of } P \} \xleftrightarrow{1:1} \{ \text{ vertices of } Q \},$$
$$\{ \text{ edges of } P \} \xleftrightarrow{1:1} \{ \text{ edges of } Q \}.$$

Finally, for each vertex of P, form a face of Q by filling in the polygon bounded by all the edges of Q that correspond to the edges of P meeting at that vertex. This gives the final bijection,

$$\{ \text{ vertices of } P \} \xleftrightarrow{1:1} \{ \text{ faces of } Q \}.$$

We say that Q is the dual of P. Repeating, the dual of Q is just a smaller copy of P sitting in Q (sitting in P...).

Exercise 1.3. If the Schläfli symbol of P is $\{p, q\}$, check that the Schläfli symbol of its dual Q is $\{q, p\}$.

Figure 1.3 shows the dual Platonic solids. We see that the tetrahedron is self-dual, the cube and the octahedron are dual to one another, as are the dodecahedron and icosahedron.

1.4 Symmetries

In order to describe the symmetry groups of the Platonic solids, we begin with some general observations. The group of symmetries of P will be denoted $W(P)$, a subgroup of $GL(\mathbb{R}^n)$.

1. If V is the set of vertices of P then the symmetries of P permute the elements in V. This defines a homomorphism $W(P) \to \mathrm{Sym}(V)$. It is easy to see that this is injective. Since $\mathrm{Sym}(V)$ is a finite group, this implies that $W(P)$ is a finite group.

2. $W(P)$ is a finite subgroup of $O(3, \mathbb{R})$; see Proposition 5.3 of the appendix.

3. This implies that $\det(g) = \pm 1$ for each $g \in W(P)$. In the case $\det(g) = 1$, $g \in SO(3, \mathbb{R})$ is a *rotation* of $\mathbb{R}^3$. We include a proof of this fact in section 5.2 of the appendix.

4. If the map $-\mathrm{Id}_{\mathbb{R}^3}$ belongs to $W(P)$ then notice that $\det(g) = -1$ if and only if $\det(-g) = 1$. This implies that $W(P) = W_0(P) \times \{\pm \mathrm{Id}_{\mathbb{R}^3}\}$, where $W_0(P) = W(P) \cap SO(3, \mathbb{R})$ is the group of rotational symmetries of P.

5. Dual solids have the same symmetry group.

Since there are only three dual pairs, (5) means that there are only three symmetry groups to compute.

The symmetries of the tetrahedron

The tetrahedron T has 4 vertices. Therefore we have an embedding $W(\mathsf{T}) \hookrightarrow \mathfrak{S}_4$, where $\mathfrak{S}_4$ is the symmetric group on four letters. However, it is clear that for any pair of vertices $\{a, b\}$ there is a reflection of T that swaps a and b and fixes the other two vertices. Since $\mathfrak{S}_4$ is generated by such transpositions, this implies that the embedding is an isomorphism. Notice that this also implies that $W(\mathsf{T})$ is generated by reflections.

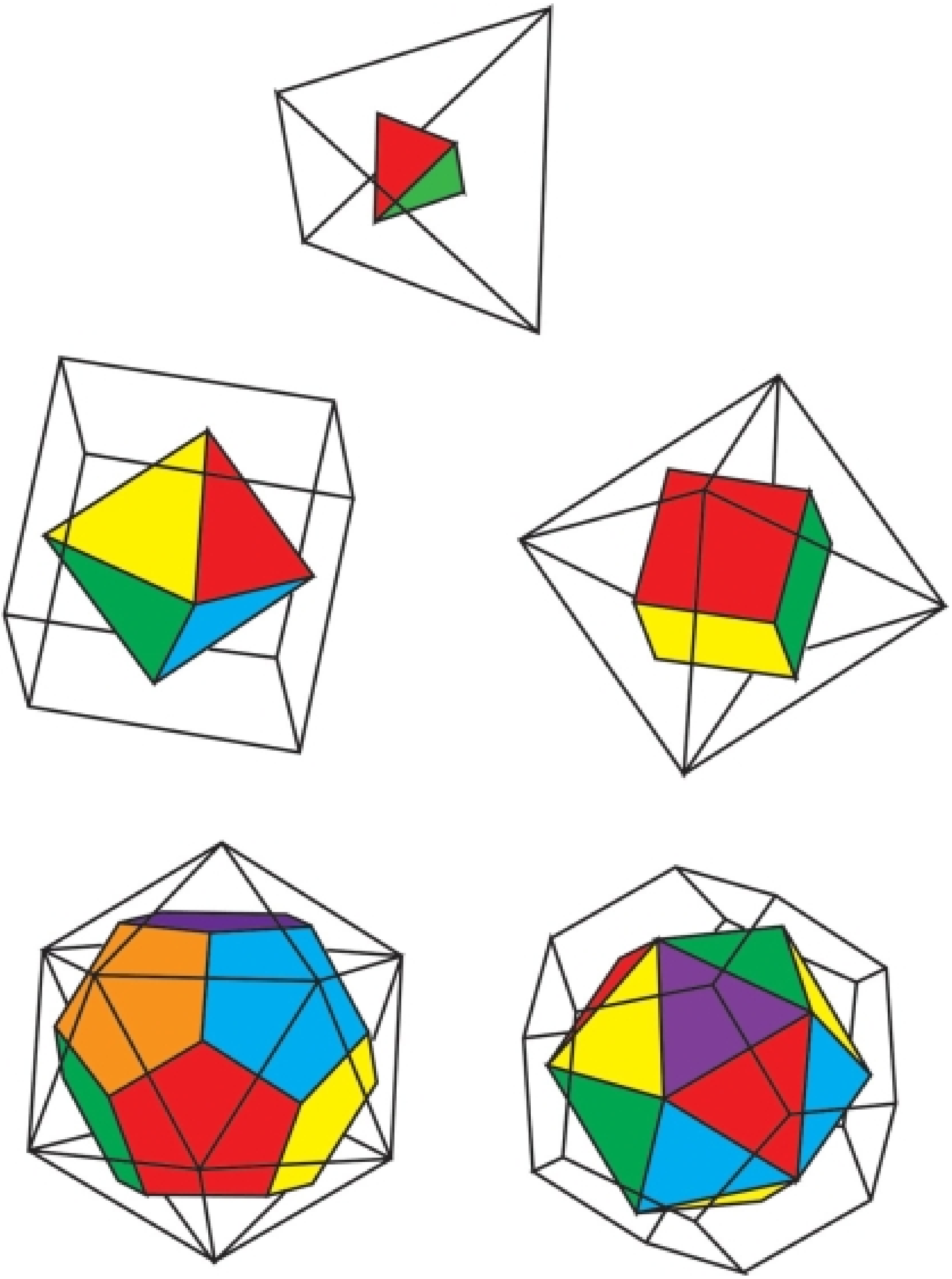


Figure 3: Dual pairs of Platonic solids.

The symmetries of the cube/octahedron

We compute the symmetries of the cube. In this case $-\mathrm{Id}_{\mathbb{R}^3}$ acts on H. Therefore, it suffices to describe the rotational symmetries of the cube. Any rotation of H will have an axis of rotation.

(A) The axis of rotation passes through the centre of a pair of opposite faces. For a given pair of faces, there are three non-trivial rotations, giving $3 \times 3 = 9$ non-trivial rotations.

(B) The axis of rotation passes through the centre of a pair of opposite edges. For a given pair of edges, there is only one non-trivial rotation, giving $12/2 = 6$ non-trivial rotations in total.

(C) The axis of rotation passes through the centre of a pair of opposite vertices. For a given pair of vertices, there are two non-trivial rotations, giving $2 \times (8/2) = 8$ non-trivial rotations.

Hence, if we remember to include the identity, we have

$$|W_0(P)| = 1 + 9 + 6 + 8 = 24, \quad |W(P)| = 48.$$

In fact, $W_0(P)$ is also isomorphic to $\mathfrak{S}_4$. To see this, let $D = \{\ell_1, \ldots, \ell_4\}$ be the set of lines connecting opposite vertices in the cube. Then $W_0(P)$ permutes the elements of D and we get a morphism $W_0(P) \to \mathfrak{S}_4$, which one can check is an isomorphism.

The symmetries of the dodecahedron/icosahedron

We compute the symmetries of the dodecahedron. As before, $-\mathrm{Id}_{\mathbb{R}^3}$ acts on D. Therefore, it suffices to describe the rotational symmetries of the dodecahedron, just as we did for the cube. Any rotation of D will have an axis of rotation.

(A) The axis of rotation passes through the centre of a pair of opposite faces. For a given pair of faces, there are four non-trivial rotations, giving $4 \times (12/2) = 24$ non-trivial rotations.

(B) The axis of rotation passes through the centre of a pair of opposite edges. For a given pair of edges, there is only one non-trivial rotation, giving $30/2 = 15$ non-trivial rotations in total.

(C) The axis of rotation passes through the centre of a pair of opposite vertices. For a given pair of vertices, there are two non-trivial rotations, giving $2 \times (20/2) = 20$ non-trivial rotations.

Hence, if we remember to include the identity, we have

$$|W_0(P)| = 1 + 24 + 15 + 20 = 60, \quad |W(P)| = 120.$$

In fact, as abstract groups, one can show that $W_0(P)$ is isomorphic to the alternating group A_5; see [6, page 50].

1.5 Reflection groups

The symmetry groups of the Platonic solids all have one key property in common. They are all examples of finite reflection groups. We denote by $O(n,\mathbb{R})$ the group of orthogonal $n \times n$ matrices $\{A \mid AA^T = 1\}$. Equivalently, this is the group of all angle and length preserving transformations of $\mathbb{R}^n$ i.e.

$$g \in O(3,\mathbb{R}) \quad \Leftrightarrow \quad (g \cdot x, g \cdot y) = (x,y) \quad \forall\, x,y \in \mathbb{R}^n.$$

A *reflection* on $\mathbb{R}^n$ is a linear map $s \in O(n,\mathbb{R})$ such that

1. $\dim \mathrm{Fix}_{\mathbb{R}^n}(s) = n-1$;
2. $s^2 = \mathrm{id}$.

Here $\mathrm{Fix}_{\mathbb{R}^n}(s) = \{x \in \mathbb{R}^n \mid s(x) = x\}$ is a vector subspace of $\mathbb{R}^n$.

Definition 1.4. A finite subgroup W of $GL(\mathbb{R}^n)$ is said to be a *reflection group* if it is generated by a set $S = \{s_1, \dots, s_k\}$ of $\mathbb{R}^n$. That is, W is the smallest subgroup of $GL(\mathbb{R}^n)$ containing all the reflections in S.

Notice that if W is a reflection group then $W \subset O(n,\mathbb{R})$ i.e. each element in W preserves both angles and lengths.

Exercise 1.5. Using the fact that $g \in W(P)$ is a reflection if and only if it has one eigenvalue equal to -1 and two eigenvalues equal to 1, count the number of reflections in $W(\mathsf{H})$ and $W(\mathsf{D})$.

We finish the lecture with a key example of a reflection group. Generalizing the tetrahedron, the n-simplex is defined to be

$$\Delta^n = \left\{(t_1, \dots, t_{n+1}) \in \mathbb{R}^{n+1} \mid t_1 + \cdots + t_{n+1} = 1,\ t_i \ge 0 \ \forall\, i\right\},$$

so that Δ^2 is the regular triangle and Δ^3 is the tetrahedron. The vertices of Δ^n are

$$e_1 = (1,0,\dots), \quad e_2 = (0,1,0,\dots), \quad \dots,$$

of which there are $n+1$ in total. Therefore the action of the symmetry group $W(\Delta^n)$ of the n-simplex on its set of vertices defines an embedding $W(\Delta^n) \hookrightarrow \mathfrak{S}_{n+1}$. On the other hand, consider the transformation $s_{i,j}$ of $\mathbb{R}^n$ which swaps the ith and jth coordinate. Clearly $s_{i,j}^2 = 1$ and the subspace $\mathrm{Fix}(s_{i,j})$ has basis $\{t_k \mid k \neq i,j\} \cup \{t_i + t_j\}$ and hence is $(n-1)$-dimensional. Thus, $s_{i,j}$ is a reflection. We see that $s_{i,j}$ just swaps the ith and jth vertex of Δ^n i.e. the image of $s_{i,j}$ in $\mathfrak{S}_{n+1}$ is the transposition (i,j). Hence, we have shown

Proposition 1.6. *The symmetry group $W(\Delta^n)$ is a reflection group, isomorphic to the symmetric group $\mathfrak{S}_{n+1}$.*

1.6 Remarks

The classical reference on Platonic solids and their symmetry groups has to be the book *Regular Polytopes*, by H.S.M. Coxeter [6]. The lecture notes [19] by Slodowy are also a fantastic introduction to these classical objects, but from the modern point of view. Finally, I should mention the superb thesis [21] by van Hoboken, where I first learned about these things. See `http://www.math.ucr.edu/home/baez/FUN.html` for a great deal more on the Platonic solids and much else besides.

2 Reflection groups and root systems

Recall that we ended the first lecture with the definition of a (finite) reflection group: a finite subgroup W of $GL(\mathbb{R}^n)$ is said to be a *reflection group* if it is generated by a set $S = \{s_1, \dots, s_k\}$ of reflections of $\mathbb{R}^n$. That is, $s_i : \mathbb{R}^n \to \mathbb{R}^n$ is a orthogonal linear map fixing a hyperplane $H \subset \mathbb{R}^n$ and $s_i^2 = 1$. This means that

$$s_i(x) = x - \frac{2(x, \alpha)}{(\alpha, \alpha)}\alpha, \tag{2}$$

for some (in fact *any*) vector α perpendicular to H. We'll call α the *root* of s_i and write $s_i = s_\alpha$.

Our presentation in this lecture is based on the book [12] by Humphreys.

2.1 Root systems

In order to classify the finite reflection groups, we introduce the notion of a root system. We fix a real vector space $\mathbb{R}^n$ with a positive definite symmetric bilinear form $(-, -)$.

Exercise 2.1. Using equation (2) show that $(w \cdot u, v) = (u, w^{-1} \cdot v)$ for all $u, v \in \mathbb{R}^n$ and $w \in W$. Hint: first check for $w = s$, a reflection, then use the fact that every element in W can be written as a product of reflections.

For simplicity, we will assume throughout that $(\mathbb{R}^n)^W = \{v \in \mathbb{R}^n \mid w \cdot v = v, \ \forall\, w \in W\}$ equals $\{0\}$. Otherwise, we simply replace $\mathbb{R}^n$ by the orthogonal compliment $V = \{v \in \mathbb{R}^n \mid (v, u) = 0, \ \forall\, u \in (\mathbb{R}^n)^W\}$ to $(\mathbb{R}^n)^W$ in $\mathbb{R}^n$. The subspace V is W-stable.

Definition 2.2. A finite subset R of $\mathbb{R}^n$ is called a *root system* if

(R0) R spans $\mathbb{R}^n$.

(R1) If $\alpha \in R$ then the only multiplies of α in R are $\pm\alpha$.

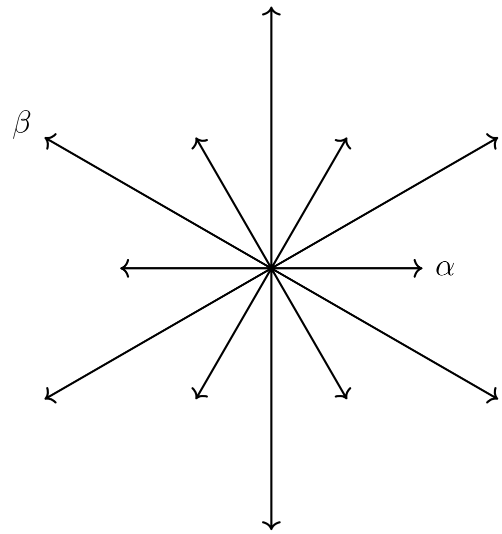
(R2) If $\alpha \in R$ then the reflection s_α maps R to itself.

The reflection group $W(R)$ associated to R is the subgroup of $GL(\mathbb{R}^n)$ generated by all $\{s_\alpha \mid \alpha \in R\}$. We give a couple of examples of root systems and the corresponding reflection groups.

Example 2.3. The dihedral groups. The dihedral group $I_2(m)$ of order $2m$ is the group of symmetries of the m-gon in $\mathbb{R}^2$. Taking every root to have length one, the corresponding root system is

$$R = \left\{ \left(\cos\frac{2k\pi}{m}, \sin\frac{2k\pi}{m} \right) \mid k = 0, 1, \dots, m-1 \right\}.$$

When $m = 6$, we get:



Here $\Delta = \{\alpha, \beta\}$ is a set of simple roots for R.

Example 2.4. The symmetric group. Let

$$E = \left\{ x = \sum_{i=1}^{n+1} x_i \epsilon_i \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0 \right\} \simeq \mathbb{R}^n,$$

where $\{\epsilon_1, \ldots, \epsilon_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$ with $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. Let $R = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n+1\}$. Then, it is an exercise to check that R is a root system for $\mathfrak{S}_n$.

Lemma 2.5. *Let R be a root system. Then the reflection group $W(R)$ of R is finite.*

Proof. By axiom (R2) of a root system every s_α maps the finite set R into itself. Therefore, there is a group homomorphism $W \to \mathfrak{S}_N$, where $N = |R|$. This map is injective: if $w \in W$ such that its action on R is trivial then, since R spas $\mathbb{R}^n$ by (R0), w acts trivially on $\mathbb{R}^n$ i.e. $w = 1$. □

Choose a vector $v \in \mathbb{R}^n$ such that $(v, \alpha) \neq 0$ for all $\alpha \in R$. Then either $\alpha \in R^+ := \{\alpha \in R \mid (v, \alpha) > 0\}$ or $\alpha \in R^- := -R^+$ i.e. we get a partition $R = R^+ \sqcup R^-$ into *positive* and *negative roots*.

Definition 2.6. Let $R = R^+ \cup R^-$ be a root system, partitioned into positive and negative roots. A *set of simple roots* for R is a subset $\Delta \subset R^+$ such that

(S1) Δ is a basis of $\mathbb{R}^n$.

(S2) Each $\beta \in R^+$ can be written as $\beta = \sum_{\alpha \in \Delta} m_\alpha \alpha$, where $m_\alpha \geq 0$ for all α.

It is also possible to define a set of simple roots, without first choosing a partition of R into positive and negative roots. Replace the axiom (S2) by

(S2') Each $\beta \in R$ can be written as $\beta = \sum_{\alpha \in \Delta} m_\alpha \alpha$, where *all* $m_\alpha \geq 0$ or *all* $m_\alpha \leq 0$.

Then a set of simple roots automatically defines a partition of R; a root β is positive if and only if all $m_\alpha \geq 0$. The problem with the above definition is that it is not at all clear that a given root system contains a set of simple roots. However, one can show:

Theorem 2.7. *Let R be a root system.*

1. *There exists a set Δ of simple roots in R.*
2. *The group W is generated by $S := \{s_\alpha \mid \alpha \in \Delta\}$.*
3. *For any two sets of simple roots Δ, Δ', there exists $w \in W$ such that $w(\Delta) = \Delta'$.*
4. *For $\alpha, \beta \in \Delta$, $\alpha \neq \beta$ we have $(\alpha, \beta) \leq 0$.*

In fact the reflection group W acts simply transitively on the collections of all sets of simple roots. The construction of all Δ's is quite easy, the difficulty is in showing that the sets constructed are indeed sets of simple roots and that the properties of Theorem 2.7 are satisfied.

Given a set of simple reflections, we call the corresponding set of reflections S a set of *simple reflections* for W. Now that we have a choice of generators for W, we can ask for a *presentation* of W i.e. a list of all relations satisfied by the reflections in S.

Theorem 2.8. *Given a set of simple reflections S, the group W has a presentation*

$$W = \langle s_\alpha : \alpha \in \Delta \mid (s_\alpha s_\beta)^{m(\alpha,\beta)} = 1 \textit{ where the integers } m(\alpha, \beta) \tag{3}$$

$$\textit{satisfy } m(\alpha, \alpha) = 1,\ m(\alpha, \beta) \geq 2 \textit{ for } \alpha \neq \beta.\ \rangle. \tag{4}$$

It's natural to ask how our presentation would change if we changed the set of generators. An arbitrary presentation might look quite different. However, if we ask that our generating set S consists solely of reflections and $|S|$ is as small as possible, then S corresponds to a set of simple reflections. Moreover, Theorem 2.7 (3) implies that the presentation we get from two different choices of simple roots is the same (up to a relabeling of the generators).

Remark 2.9. Groups that have a presentation as in (4) are called *Coxeter groups* since they were extensively studied by him. We will see below the classification of finite Coxeter groups. However, there is also a whole zoo of infinite Coxeter groups out there, which play a key role in several areas of mathematics, such as geometric group theory.

Example 2.10. If W is the dihedral group $I_2(m)$, then we can deduce from example 2.3 that

$$I_2(m) = \langle s_\alpha, s_\beta \mid s_\alpha^2 = s_\beta^2 = (s_\alpha s_\beta)^m = 1 \rangle.$$

Similarly, for the symmetric group, example 2.4 implies that

$$\mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1,\ (s_i s_j)^2 = 1,\ \forall |i-j| > 1,\ (s_i s_{i+1})^3 = 1,\ i = 1, \dots, n-2 \rangle$$

The relations $(s_i s_j)^2 = 1$ and $(s_i s_{i+1})^3 = 1$ can be rewritten as

$$s_i s_j = s_j s_i,\ \forall |i-j| > 1, \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ \forall\ i = 1, \dots, n-2.$$

They are called the *braid relations* since they are the defining relations of the braid group $\mathcal{B}_n$, of which $\mathfrak{S}_n$ is a quotient.

2.2 Coxeter graphs and symmetric forms

So far we have used the notion of root system and simple roots to show that a finite reflection group has a very particular presentation. Our approach to classifying these groups will be to try and classify the finite groups that have these particular presentations. In order to do this, we need some background on symmetric forms and symmetric matrices.

A symmetric bilinear form $(-,-)$ on $\mathbb{R}^n$ is the same[1] as a real $n \times n$-matrix A such that $A^T = A$ i.e. A is a *symmetric matrix*. Explicitly, given a form $(-,-)$, we fix a basis $\epsilon_1, \dots, \epsilon_n$ of $\mathbb{R}^n$ and set

$$a_{i,j} := (\epsilon_i, \epsilon_j),$$

so that $(v,w) = v^T A w$ for all $v, w \in \mathbb{R}^n$.

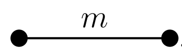
Definition 2.11. A symmetric real matrix A is said to be

- *positive definite* if $x^T A x > 0$ for all $0 \neq x \in \mathbb{R}^n$.
- *positive semi-definite* if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$.

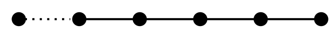
Before we relate reflection groups to symmetric matrices, we will first construct a *Coxeter graph* out of W. Then there is a simple algorithm to associate to the Coxeter graph a symmetric bilinear form (or rather, the algorithm shows that the Coxeter graph "remembers" the fact that

[1] Rather, it is the same as a real symmetric matrix *once* a basis of $\mathbb{R}^n$ has been chosen.

it came from a reflection group acting on $\mathbb{R}^n$, with its standard form). So we fix a set of simple roots Δ for R. The vertices of the Coxeter graph Γ are labeled by the corresponding simple roots / simple reflections. Associated to any pair $\alpha, \beta \in \Delta$, we have the integer $m(\alpha, \beta) \geq 2$, as in Theorem 2.8. If $m(\alpha, \beta) = 2$ then there is no edge between α and β. Otherwise, we add an edge between α and β, decorated with the number $m(\alpha, \beta)$. Since $m(\alpha, \beta) = 3$ occurs frequently, we don't bother to label the edge in this case.

Example 2.12. Going back to our examples of the dihedral and symmetric groups, example 2.10 implies that the Coxeter graph for $I_2(m)$ is 

. For the symmetric group $\mathfrak{S}_{n+1}$, we get the Coxeter group of type A_n,



with n vertices.

In general a Coxeter graph is a finite graph Γ with at most one edge between two vertices, no loops (i.e. an edge from a vertex to itself) and an integer $m \geq 3$ labeling each edge. Next, if Γ has n vertices, then we associated to it a symmetric bilinear form on $\mathbb{R}^n$, or equivalently a symmetric real matrix A. Let $A = (a_{\alpha,\beta})$, where $a_{\alpha,\alpha} = 1$ and

$$a_{\alpha,\beta} = -\cos \frac{\pi}{m(\alpha, \beta)}, \quad \forall\, \alpha \neq \beta.$$

Proposition 2.13. *Assume that the real symmetric matrix A comes from a finite reflection group as above. Then A is positive definite.*

Proof. Let $(-,-)$ denote the usual symmetric bilinear form on $\mathbb{R}^n$ such that the usual basis $\{\epsilon_1, \dots, \epsilon_n\}$ is orthonormal. Then it is clearly positive definite. The matrix A doesn't depend of the length of the simple roots so we may assume without loss of generality that $\|\alpha\| = 1$ for all $\alpha \in \Delta$. As noted in section 2.5 below, $(\alpha, \beta) = \|\alpha\| \|\beta\| \cos \theta$, where θ is the angle between α and β. Theorem 2.7 (4) implies that $\theta \geq \pi$. We want to express θ in terms of $m(\alpha, \beta)$. The key is the fact that $s_\alpha s_\beta$ is rotation by twice θ (see the exercise sheet). This implies that $\theta = \pi \left(\frac{1}{m(\alpha,\beta)} + 1 \right)$ and hence

$$(\alpha, \beta) = \cos \pi \left(\frac{1}{m(\alpha, \beta)} + 1 \right) = -\cos \frac{\pi}{m(\alpha, \beta)}.$$

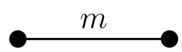
This means that $\alpha^T A \beta = (\alpha, \beta)$ for all $\alpha, \beta \in \Delta$ i.e. A is just the standard form $(-,-)$ expressed in the basis of $\mathbb{R}^n$ given by the simple roots Δ. In particular, it is positive definite. □

What we will see from the classification is that the finite reflection groups, up to isomorphism, are in bijection with the Coxeter graphs whose associated symmetric matrix A is positive definite.

For brevity, we will say that Γ is a positive (semi-)definite Coxeter graph if the corresponding matrix A is positive (semi-)definite.

Exercise 2.14. Show that the symmetric matrix

$$\begin{pmatrix} 1 & -\cos\frac{\pi}{m} \\ -\cos\frac{\pi}{m} & 1 \end{pmatrix}$$

corresponding to the Coxeter graph 

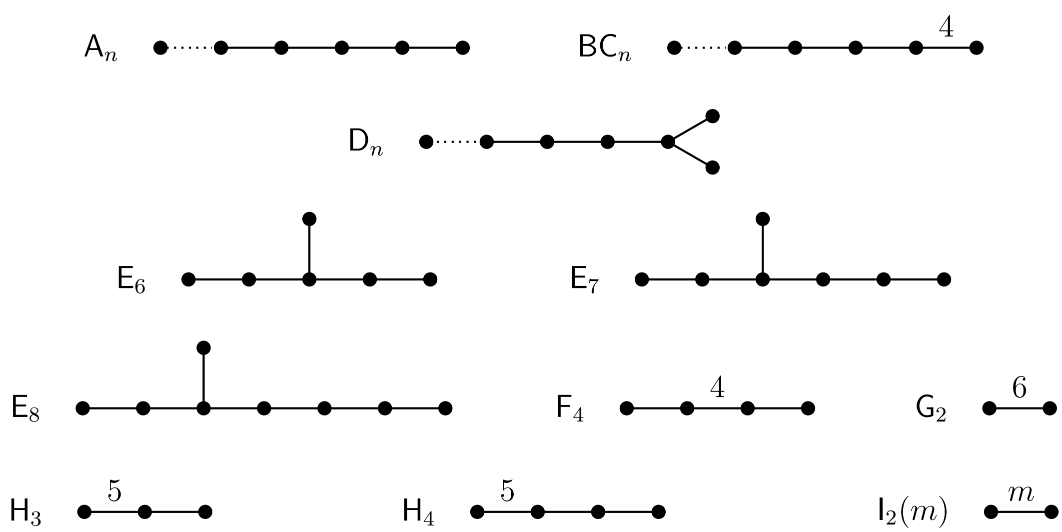
is positive definite.

2.3 The classification

There are two steps in the classification of positive definite Coxeter graphs. First, we guess all positive definite and semi-definite graphs. A direct calculation shows whether a particular graph is positive (semi-)definite. Then we use this list, together with a key result from linear algebra to prove that there are no others.

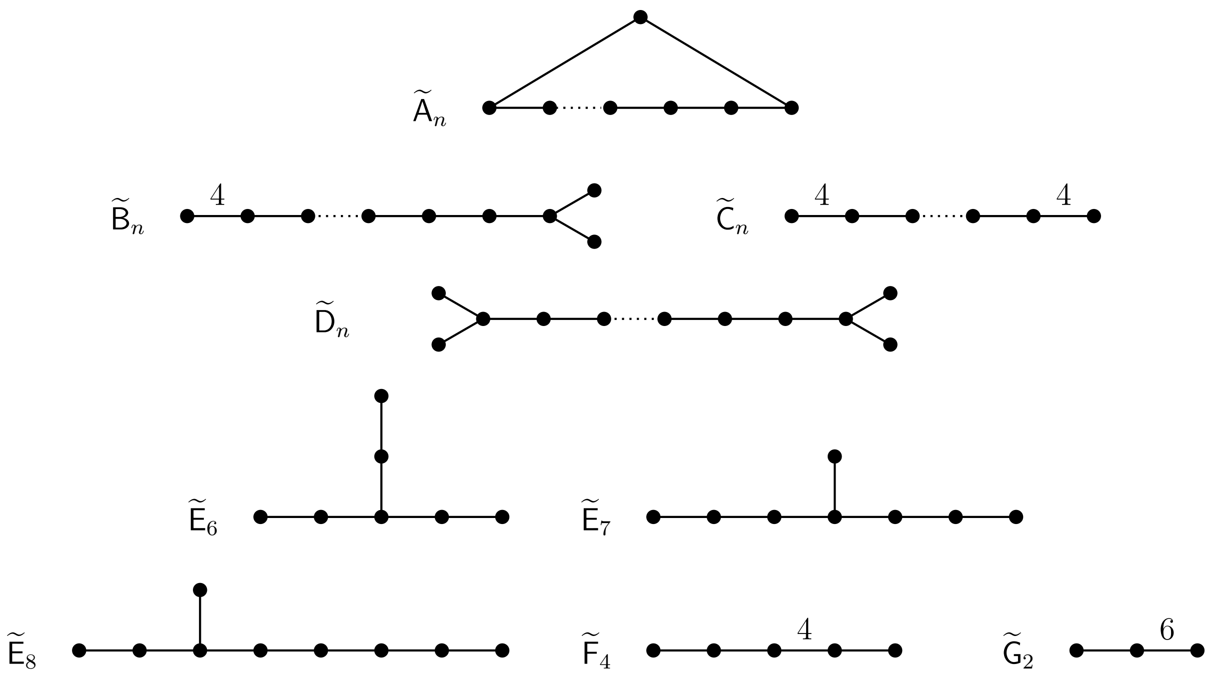
The proof of the following proposition is a direct calculation. By induction, one calculates the eigenvalues of the associated symmetric matrix A (give it a go!).

Proposition 2.15. *The following Coxeter graphs are positive definite.*



Similarly, one checks by a direct calculation:

Proposition 2.16. *The following Coxeter graphs are all positive semi-definite.*



We recall the key "Spectral Theorem for real symmetric matrices". A proof of the theorem can be found in the appendix.

Theorem 2.17. *Let A be a real valued symmetric matrix.*

1. *The eigenvalues of A are real.*
2. *There exists an orthogonal matrix $g \in \mathrm{O}(n, \mathbb{R})$ such that gAg^T is diagonal.*

Example 2.18. The spectral theorem for real symmetric matrices implies that a real symmetric matrix is positive (semi-)definite if and only if all its eigenvalues are > 0 (≥ 0). As an example, consider the Coxeter graph of type $\widetilde{F}_4$. The corresponding symmetric matrix is

$$A = \begin{pmatrix} 1 & \frac{-1}{2} & 0 & 0 & 0 \\ \frac{-1}{2} & 1 & \frac{-1}{2} & 0 & 0 \\ 0 & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} & 1 & \frac{-1}{2} \\ 0 & 0 & 0 & \frac{-1}{2} & 1 \end{pmatrix}.$$

A rather lengthy computation shows that the eigenvalues of A are $0, \frac{1}{2}, 1, \frac{3}{2}$ and 2. Therefore $\widetilde{F}_4$ is positive semi-definite.

A matrix A is said to be *decomposable* if, after a permutation of rows and columns, it is the sum of two blocks

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

otherwise we say that A is *indecomposable*. We will use the spectral theorem to prove the following result in linear algebra. The proof of the following key proposition is based on Theorem 5.1. The proof can be found on [12, page 35].

Proposition 2.19. *Let A be a symmetric real matrix which is*

(a) positive semi-definite,

(b) indecomposable,

(c) satisfies $a_{i,j} \leq 0$ for all $i \neq j$.

Then the radical

$$\operatorname{rad} A = \{v \in \mathbb{R}^n \mid v^T A v = 0\}$$

of A equals the kernel of A, and is one-dimensional. Moreover, the smallest eigenvalue of A has multiplicity one, and the corresponding eigenvector can be chosen to have all strictly positive coordinates.

Corollary 2.20. *If Γ is a connected semi-positive definite Coxeter graph, then every proper subgraph is a positive definite Coxeter graph.*

Proof. If Γ is connected then the matrix A is indecomposable and clearly satisfies the other conditions of Proposition 2.19. Let Γ' be a subgraph of Γ and A' the corresponding symmetric matrix. If $|\Gamma'| = k \leq |\Gamma| = n$, then A' is a $k \times k$ matrix. Then $m'(\alpha, \beta) \leq m(\alpha, \beta)$, which implies that $a'_{\alpha,\beta} = -\cos\frac{\pi}{m'(\alpha,\beta)} \geq a_{\alpha,\beta} = -\cos\frac{\pi}{m(\alpha,\beta)} = a_{\alpha,\beta}$. We will assume that Γ' is not positive definite (in particular, Γ' is a proper subgraph of then there exists some non-zero vector $v \in \mathbb{R}^k$ such that $v^T A' v \leq 0$. If $v' = (|v_1|, \ldots, |v_k|, 0, \ldots, 0)$ then

$$\begin{aligned} 0 &\leq \sum_{\alpha,\beta} a_{\alpha,\beta}|v_\alpha||v_\beta| \leq \sum_{\alpha,\beta} a'_{\alpha,\beta}|v_\alpha||v_\beta| \\ &\leq \sum_{\alpha,\beta} a'_{\alpha,\beta} v_\alpha v_\beta \leq 0. \end{aligned}$$

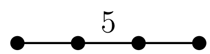
Here the sum is over all $\alpha, \beta \leq k$ and we have used the fact that $a_{\alpha,\beta} \leq 0$ if $\alpha \neq \beta$. Therefore the inequalities are equalities. Moreover, we see that $(v')^T A v' = 0$. Therefore Proposition 2.19

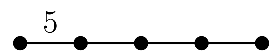
implies that v' is in the kernel of A. But again, Proposition 2.19 implies that $k = n$ and each v_i is non-zero. However, the other inequalities now imply that $a'_{\alpha,\beta} = a_{\alpha,\beta}$ for all α, β i.e. $A' = A$ and hence $\Gamma' = \Gamma$, contradicting the fact Γ' is a proper subgraph of Γ. □

Theorem 2.21. *Let Γ be a finite connected graph. The positive definite Coxeter forms listed in Proposition 2.15 constitute all such forms.*

Proof. The key to the proof of this theorem is Corollary 2.20. By Corollary 2.20, if we are given a positive semi-definite graph then every proper graph is positive definite. We will show that every positive semi-definite graph Γ is contained in the list given in Proposition 2.16. It will then follow that the list given in Proposition 2.15 contains all positive definite Coxeter graphs.

We will need the following fact. An easy calculation of eigenvalues shows that

Z_4 

Z_5 

are not positive (semi-)definite graphs.

So we assume that Γ is a positive semi-definite Coxeter graph. Recall that the *valence* of a vertex is the number of edges leaving the vertex.

(a) Let us consider first cycles in Γ. If Γ contains a cycle of length at least three, then it contains $\widetilde{\mathsf{A}}_n$ for some n. Hence it must equal $\widetilde{\mathsf{A}}_n$. So we assume Γ contains at most 2-cycles.

(b) If Γ contains a pair of vertices with ≥ 5 edges (or in our presentation, this is an edge labeled by $m \geq 7$) between them then the fact that it cannot contain $\widetilde{\mathsf{G}}_2$ implies that $\Gamma = I_2(m)$ for some m. But then it is not positive semi-define, so there are at most 4 edges between vertices.

(c) If Γ contains an edge of weight 6, then there must be at least three vertices in Γ or else $\Gamma = I_2(6)$. But if it contains at least three vertices it contains a copy of $\widetilde{\mathsf{G}}_2$ (and must equal $\widetilde{\mathsf{G}}_2$). Hence we may assume that every edge of Γ has weight at most 5.

(d) Assume that Γ has an edge of weight 5. If it has only three vertices then the other edge cannot be of weight 4 since Γ would contain a copy of $\widetilde{\mathsf{B}}_2$. So this edge must have weight 3. But then $\Gamma = \mathsf{H}_3$ is positive definite. Thus, Γ must have at least 4 vertices.

(e) The fact that Γ cannot contain Z_4 implies that the unweighted graph is of type A such that 5 labels the left most edge.

(f) The fact that Γ cannot contain Z_5 implies that Γ has exactly 4 vertices.

(h) The fact that Γ cannot contain $\widetilde{\mathsf{C}}_3$ implies that Γ must in fact be H_4. But then Γ is positive definite. So we conclude that there are no positive semi-define Coxeter graphs with an edge of weight 5.

(i) Thus, we are reduced to the situation where every edge of Γ has weight at most 4. If Γ contains at least two edges of weight 4 then it contains $\widetilde{\mathsf{C}}_n$. So we may assume that Γ contains at most one edge of weight 4.

(j) If the unweighted graph of Γ contains a vertex of valence at least 3 then it contains a copy of $\widetilde{\mathsf{B}}_n$. Therefore we may assume that the unweighted graph of Γ is of type A.

(k) If Γ has at least 5 vertices then it contains a copy of $\widetilde{\mathsf{F}}_4$ or is of type BC_n, so it must equal $\widetilde{\mathsf{F}}_4$.

(l) If Γ contains fewer than 5 vertices then it is either F_4 or type BC_n i.e. it is not positive semi-definite. Hence we are finally reduced to the case where every edge has weight 3.

(m) If Γ has a vertex of valency at least 4 then it contains a copy of $\widetilde{\mathsf{D}}_4$. So we may assume that each vertex has valency at most 3.

(n) If it contains at least two vertices of valency 3 then it contains a copy of $\widetilde{\mathsf{D}}_n$. Therefore we may assume it contains exactly one vertex of valency 3.

(o) Then $\Gamma = T(p, q, r)$ is the spoke with arms of length p, q, r (so that Γ has $p + q + r$ edges in total).

(p) At least one of p, q, r is equal to 1 otherwise Γ contains a copy of $\widetilde{\mathsf{E}}_6$ or is of positive definite type. So without loss of generality, $p = 1$.

(p) If $q, r \geq 3$ then Γ contains a copy of $\widetilde{\mathsf{E}}_7$. Therefore we may assume without loss of generality that $q = 2$.

(q) If $r \geq 4$ then Γ contains a copy of $\widetilde{\mathsf{E}}_8$, otherwise it is of positive definite type.

The fact that the positive definite graphs listed in Proposition 2.15 are all possible ones is proved in a similar fashion, but the proof is much simpler. We leave it to the interested reader. $\square$

We haven't quite completed the classification of finite reflection groups. One still has to construct an explicit reflection group for each of the positive definite Coxeter graphs of Proposition 2.15. For the infinite series $\mathsf{A}_n, \mathsf{BC}_n$ and D_n this is straight-forward (we have seen type A already, type BC corresponds to the hyperoctahedral group and D is the normal subgroup of BC_n of index 2 consisting of all signed permutation matrices such that the product of the non-zero entries is one), but the exceptional reflection groups of type E, F and H are rather difficult to construct. We have actually seen the reflection group H_3 already - this is a reflection group acting on $\mathbb{R}^3$, it is none other than $W(\mathsf{D})$, the symmetry group of the dodecahedron.

2.4 Crystallographic groups

Most reflection groups W have the additional property that they preserve a lattice $L \subset \mathbb{R}^n$ i.e. $s_\alpha(L) \subset L$ for all $\alpha \in \Delta$. Here a *lattice* is the $\mathbb{Z}$-span $\mathbb{Z} \cdot \beta_1 + \cdots \mathbb{Z} \cdot \beta_n$ of some basis $\{\beta_1, \dots, \beta_n\}$ of $\mathbb{R}^n$. In this case we say that W is a *crystallographic reflection group*.

Lemma 2.22. *If W is crystallographic then $m(\alpha, \beta) = 2, 3, 4$ or 6 for all $\alpha \neq \beta$.*

Proof. Assume that W preserves a lattice $L = \mathbb{Z} \cdot \beta_1 + \cdots \mathbb{Z} \cdot \beta_n$. Then, any element $w \in W$, written as a matrix with respect to the basis $\{\beta_1, \dots, \beta_n\}$ has trace in $\mathbb{Z}$. Now, $s_\alpha s_\beta$ is rotation by $\frac{2\pi}{m(\alpha,\beta)}$ (see exercise 3 (c)), hence $\mathrm{Tr}(s_\alpha s_\beta) = (n-2) + 2\cos\frac{2\pi}{m(\alpha,\beta)}$ which implies that $\cos\frac{2\pi}{m(\alpha,\beta)} \in \frac{1}{2}\mathbb{Z}$. This forces $m(\alpha, \beta) = 2, 3, 4$ or 6. □

Based on this simple observation, it is natural to extend the definition of root system.

Definition 2.23. Let R be a root system. Then R is said to be *crystallographic* if, in addition to axioms (R1) and (R2) of definition 2.2, we have

(R3) If $\alpha, \beta \in R$ then

$$n_{\beta,\alpha} := \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

belongs to $\mathbb{Z}$.

We can easily check from the classification of Theorem 2.21 which finite reflection groups have the property that $m(\alpha, \beta) = 2, 3, 4$ or 6 for all $\alpha \neq \beta$. These are the ones of type $\mathsf{A}, \mathsf{BC}, \mathsf{D}, \mathsf{E}, \mathsf{F}$ and G. Then, for each of these, one can explicitly construct a crystallographic root system. It turns out that for all types, except BC, there is only one crystallographic root system of that type. However, for the reflection group of type BC, there are two different crystallographic root systems, which we call type B and type C. This construction implies:

Proposition 2.24. *If W is crystallographic then it admits a crystallographic root system.*

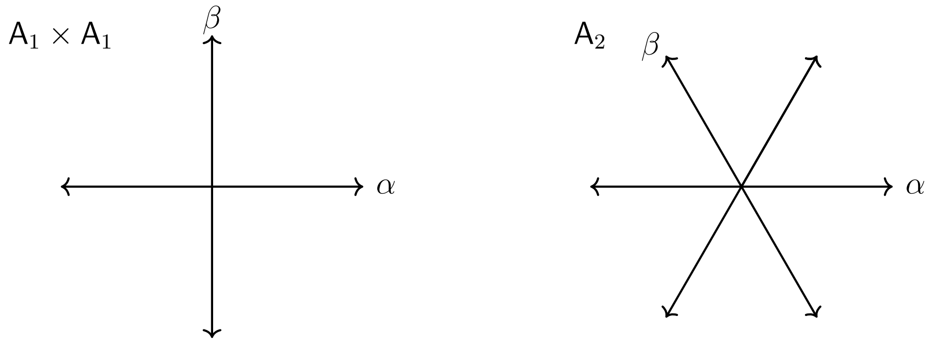


Figure 4: The crystallographic root systems of type $A_1 \times A_1$ and A_2.

2.5 The angle between roots

Recall that if $\alpha, \beta \in E$ are non-zero vectors, then the angle θ between α and β can be calculated using the cosine formula $||\alpha|| \cdot ||\beta|| \cos\theta = (\alpha, \beta)$. Thus,

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = 2\frac{||\beta||}{||\alpha||}\cos\theta,$$

and hence $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4\cos^2\theta$. If $\alpha, \beta \in R$, then the fourth axiom implies that $4\cos^2\theta$ is a positive integer. Since $0 \leq \cos^2\theta \leq 1$, we have $0 \leq \langle \beta, \alpha \rangle \langle \alpha, \beta \rangle \leq 4$. Hence the only possible values of $\langle \alpha, \beta \rangle$ are:

$\langle \beta, \alpha \rangle$	$\langle \alpha, \beta \rangle$	θ
0	0	$\frac{\pi}{2}$
1	1	$(\star)$
-1	-1	$\frac{2\pi}{3}$
1	2	$(\star\star)$
-1	-2	$\frac{3\pi}{4}$
1	3	$\frac{\pi}{6}$
-1	-3	$(\star\star\star)$

(5)

From this, it is possible to describe the rank two root systems. They are of type $A_1 \times A_1, A_2, B_2$ and G_2. See Figures 2.5 and 2.5. In each of the diagrams, $\{\alpha, \beta\}$ form a set of simple roots.

2.6 The surfer dude and E_8

The crystallographic root system of type E_8, which is by far the most complicated of all the root systems, made world headlines in 2007. First, there was the story of how a group of mathematicians

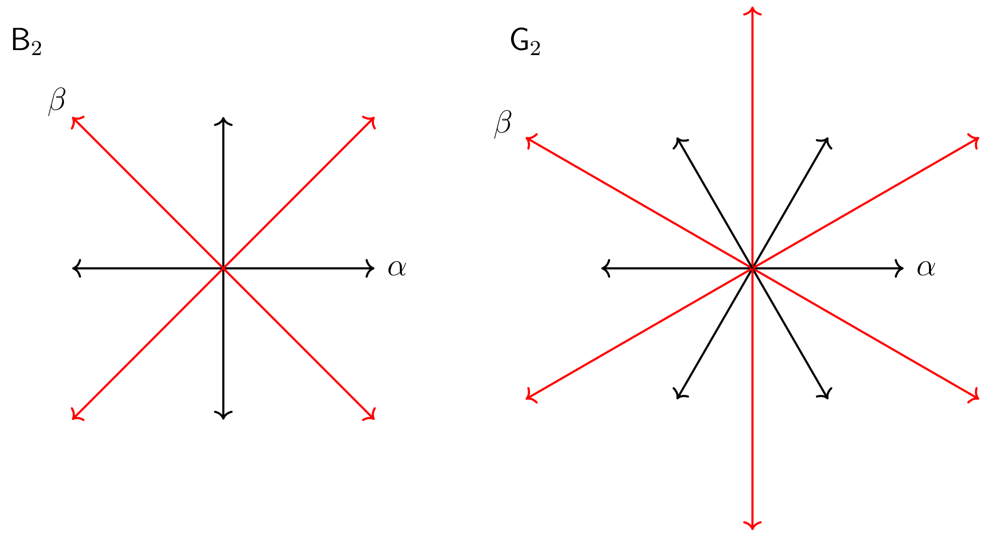


Figure 5: The crystallographic root systems of type B_2 and G_2. The long roots are coloured red and the short roots are black.

had finally computed the "Kazhdan-Lusztig-Vogan polynomials" for the reflection group of E_8 (the group $W(\mathsf{E}_8)$ has order 696729600), using some clever computer algorithms and a huge number of computers; the computation was even picked up in the main-stream media. See

`http://aimath.org/E8/`
`http://www.aimath.org/E8/computerdetails.html`

For details on the mathematics involved, check out

`http://www-math.sp2mi.univ-poitiers.fr/~maavl/pdf/Nieuw-Archief.pdf`

too.

Then, a short while later, a paper appeared on the internet by Garrett Lisi, claiming that the physical "Grand Unifying Theory of Everything" i.e. the ultimate theory of physics, which would finally combine general relativity with quantum mechanics, could be constructed using the representation theory of the Lie group of type E_8. This outlandish idea actually seemed to have some physical and mathematical merit and was intensely studied by both groups for a short while. What made the story even more interesting was that Lisi was actually a professional surfer, based in Hawaii. Though he had a PhD in physics, he had left academia for surfing. As far as I am aware his idea didn't work out in the end, but you can dig into this yourself, starting with

`http://www.telegraph.co.uk/news/science/large-hadron-collider/3314456/`

Surfer-dude-stuns-physicists-with-theory-of-everything.html
https://en.wikipedia.org/wiki/An_Exceptionally_Simple_Theory_of_Everything
http://www.scientificamerican.com/article/garrett-lisi-e8-theory/

Finally, the beautiful Figure 2.6 is the projection of the E_8 root system (which lives in $\mathbb{R}^8$) onto the plane.

2.7 Invariant Theory

Finite reflection groups appear in many situations in mathematics (and physics). One of the most classical appearances is in invariant theory. Let $W \subset GL(\mathbb{R}^n)$ be a group. Then the group W also acts naturally on the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ by $(g \cdot f)(a) = f(g^{-1} \cdot a)$ for all $f \in \mathbb{R}[x_1, \ldots, x_n]$, $g \in W$ and $a \in \mathbb{R}^n$. The ring $\mathbb{R}[x_1, \ldots, x_n]^W$ of functions invariant under W is a very complicated ring in general. The study of these rings is called *invariant theory*. One of the cornerstones of the theory is the Chevalley Theorem.

Theorem 2.25. *Assume $W \subset GL(\mathbb{R}^n)$ is finite. Then $\mathbb{R}[x_1, \ldots, x_n]^W$ is a polynomial ring if and only if W is a reflection group.*

Example 2.26. We can make the symmetric group $\mathfrak{S}_n$ act on $\mathbb{R}^n$ by permuting the coordinates. We've seen that this makes $\mathfrak{S}_n$ into a reflection group. Then $\mathbb{R}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ is the ring of symmetric polynomials. It is a classical result (attributed to Newton) that

$$\mathbb{R}[x_1, \ldots, x_n]^{\mathfrak{S}_n} = \mathbb{R}[e_1, \ldots, e_n],$$

where

$$e_r(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r}$$

is the rth *elementary symmetric polynomial.*

Exercise 2.27. Take $\mathbb{Z}_2 = \{1, s\}$ acting on $\mathbb{R}^2$ by $s \cdot x_1 = -x_1$ and $s \cdot x_2 = -x_2$. Show that $\mathbb{R}[x_1, x_2]^{\mathbb{Z}_2}$ is *not* a polynomial ring.

2.8 Lie algebras

Finite crystallographic reflection groups, described via root systems, first appeared in a (seemingly) very different context.

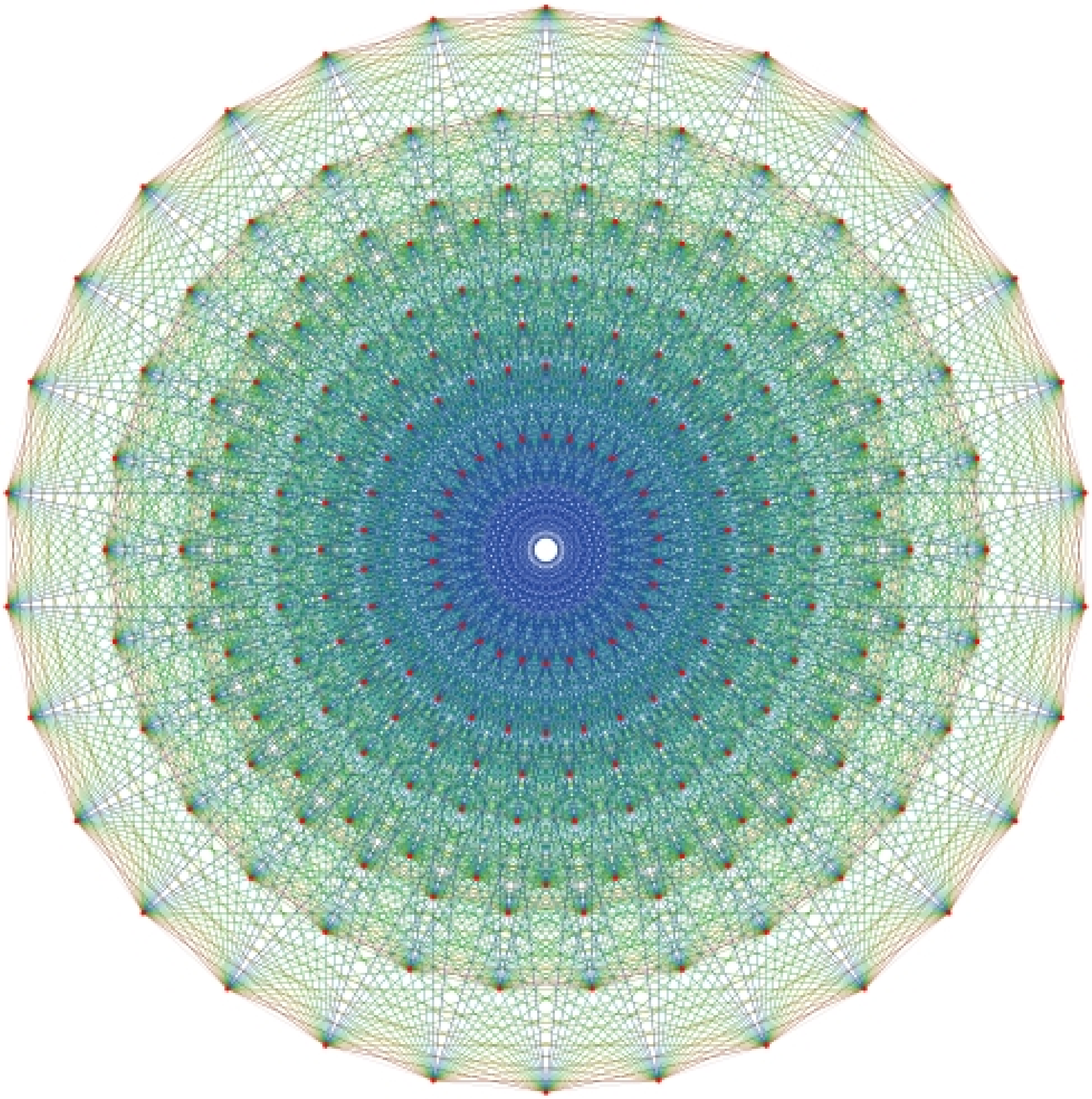


Figure 6: The root system of E_8.

At the opposite end of the spectrum to reflection groups are groups of continuous transformations; for instance, the group of all possible rotations of the plane. These groups have an additional structure, namely the set of elements in the group is a smooth space or *manifold* and the multiplication map $G \times G \to G$ is a differentiable map (as is the inverse map). These groups are called *Lie groups*, after the ground breaking work of the Norwegian mathematician Sophus Lie. These groups are often very complicated as spaces. Therefore, in order to understand them better, the standard approach is to replace them by a linearization - a vector space approximation of G, with some additional structure that "remembers" some of the structure of the group. This additional structure on the vector space $\mathfrak{g}$ is a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that makes $(\mathfrak{g}, [-,-])$ into a *Lie algebra*. Formally, the axioms of a Lie algebra are

(L1) The bracket $[-,-]$ is *anti-symmetric* i.e.

$$[X, Y] = -[Y, X], \quad \forall\ X, Y \in \mathfrak{g}.$$

(L2) The bracket $[-,-]$ satisfies the *Jacobi identity* i.e.

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0, \quad \forall\ X, Y, Z \in \mathfrak{g}.$$

What does all this have to do with root systems? Lie algebras are "built up", in a precise sense, from the simple Lie algebras. Remarkably, one can associate, in a very natural way, to every simple Lie algebra a crystallographic root system. It was shown by E. Cartan and Killing that a Lie algebra is uniquely defined by its root system. Thus, the classification of simple Lie algebras is equivalent to the classification of root systems.

The typical example of a Lie algebra is the space $\mathfrak{g} = \mathrm{Mat}_n(\mathbb{R})$ of all $n \times n$ real matricies. The Lie bracket on $\mathfrak{g}$ is the commutator bracket $[A, B] := AB - BA$. This is the Lie algebra associated with the Lie group $GL(\mathbb{R}^n)$ of all invertible linear maps $\mathbb{R}^n \to \mathbb{R}^n$; or in more down to earth terms, the group of all invertible $n \times n$ real matricies.

Exercise 2.28. Check that $(\mathrm{Mat}_n(\mathbb{R}), [-,-])$ satisfies the axioms of a Lie algebra.

Lie groups and Lie algebras come under the general heading of *Lie theory*. It is a beautiful and incredibly rich theory, which I would strongly recommend to everyone.

2.9 Remarks

There is a huge body of literature on the very classical subject of finite reflection groups and general Coxeter groups. Two standard references are the books *Reflection Groups and Coxeter*

Groups, by Humphreys [12], and *Geometry of Coxeter groups*, by Hiller [10]. A quick google will reveal thousands more references.

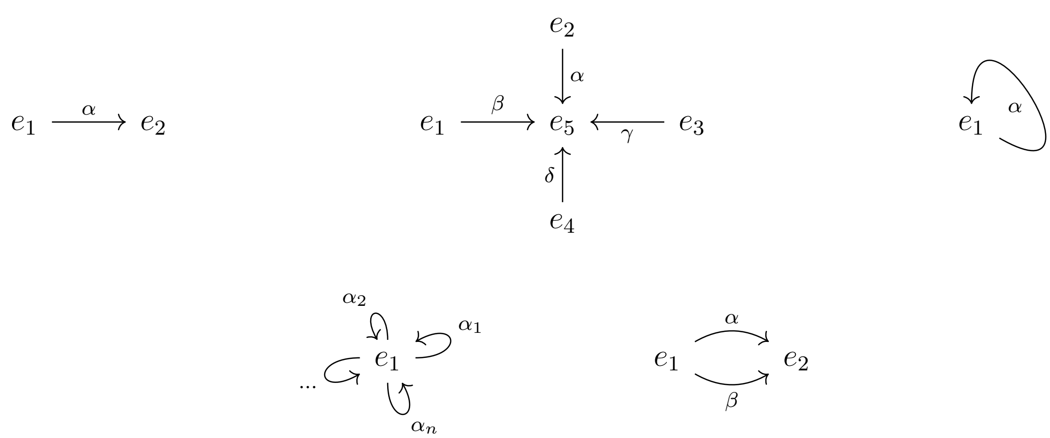
3 Quivers

In the finally two lectures, we turn instead to quivers and their representations. Though these don't seem to have anything to do with reflection groups, we'll see in the final lecture on Gabriel's Theorem that reflection groups and their root systems play a key role in the representation theory of quivers.

3.1 What is a quiver?

A *quiver* Q is simply a directed graph. That is, it is a collection $Q_0 = \{e_1, \ldots, e_k\}$ of vertices and a collection $Q_1 = \{\alpha_1, \ldots, \alpha_\ell\}$ of arrows between vertices. In order to specify which two vertices an arrow α goes between, we define a pair of functions $h, t : Q_1 \to Q_0$ such that $h(\alpha)$ is the vertex at the *head* of α and $t(\alpha)$ is the *tail* of α i.e. α is the arrow beginning at vertex $t(\alpha)$ and ending at vertex $h(\alpha)$.

Example 3.1. Here are some examples of quivers.



For instance, in the second example, $Q_0 = \{e_1, e_2, e_3, e_4, e_5\}$, $Q_1 = \{\alpha, \beta, \gamma, \delta\}$ and $t(\gamma) = e_3$, $h(\beta) = e_5$ etc.

Remark 3.2. A quiver is the case, holdall, or tube that an archer uses to hold his/her arrows.

3.2 Representations

As representation theorists, we're interested in *representations* of a quiver. A representation $M = \{(V_i, \varphi_\alpha) \mid i \in Q_0,\ \alpha \in Q_1\}$ is the data of a complex vector space V_i at each vertex i of Q

and a linear map $\varphi_\alpha : V_{t(\alpha)} \to V_{h(\alpha)}$ between the vector space at the tail of α to the vector space at the head of α, for every arrow α.

Example 3.3. Here are two representations

$$M: \qquad \mathbb{C}^2 \overset{\varphi_\alpha}{\underset{\varphi_\beta}{\rightrightarrows}} \mathbb{C}^3, \qquad\qquad N: \qquad \mathbb{C}^4 \overset{\phi_\alpha}{\underset{\phi_\beta}{\rightrightarrows}} \mathbb{C} \tag{6}$$

where

$$\varphi_\alpha = \begin{pmatrix} 4 & 2 \\ 3 & 3 \\ 0 & 9 \end{pmatrix}, \quad \varphi_\beta = \begin{pmatrix} -4 & 34 \\ 0 & 0 \\ 12 & \pi \end{pmatrix}, \quad \phi_\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}, \quad \phi_\beta = \begin{pmatrix} 4 & 1 & -101 & 19 \end{pmatrix}.$$

We can associate to a representation $M = \{(V_i, \varphi_\alpha)\}$ a *dimension vector* $\underline{\dim}\ M \in \mathbb{Z}^{|Q_0|}$ by setting

$$(\underline{\dim}\ M)_i = \dim V_i.$$

Unlike a vector space, the representation M is not uniquely defined by its dimension vector. However, as we shall see, $\underline{\dim}\ M$ is a very useful invariant of M.

Example 3.4. If we consider the two representations M and N in (6), then

$$\underline{\dim}\ M = (2, 3), \quad \underline{\dim}\ N = (4, 1).$$

3.3 Subrepresentations, simple representations etc.

Given a representation $M = \{(V_i, \varphi_\alpha) \mid i \in Q_0,\ \alpha \in Q_1\}$ of Q, a *subrepresentation* N of M is the representation $\{(U_i, \varphi_\alpha|_{U_{t(\alpha)}}) \mid i \in Q_0,\ \alpha \in Q_1\}$, where each U_i is a subspace of V_i, chosen so that $\varphi_\alpha(U_{t(\alpha)}) \subset U_{h(\alpha)}$ for all $\alpha \in Q_1$.

Example 3.5. Here is an example of a subrepresentation. Let M be the representation of $e_1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} e_2$ given in (6). Let $U_1 = \mathbb{C}$ be the subspace of $\mathbb{C}^2$ spanned by $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $U_2 = \mathbb{C}^2$ the subspace of $\mathbb{C}^3$ spanned by

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then, M restricts to the subrepresentation $U_1 \overset{\varphi'_\alpha}{\underset{\varphi'_\beta}{\rightrightarrows}} U_2$, where

$$\varphi'_\alpha = \begin{pmatrix} 2 \\ -9 \end{pmatrix}, \quad \varphi'_\beta = \begin{pmatrix} -38 \\ 12-\pi \end{pmatrix}.$$

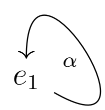
If on the other hand, we take the representation N in (6) and U_1 the subspace of $\mathbb{C}^4$ spanned by the vectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and $U_2 = \{0\} \subset \mathbb{C}$, then this does *not* produce a subrepresentation of N since

$$\phi_\alpha(U_1) \not\subset U_2, \quad \phi_\beta(U_1) \not\subset U_2,$$

i.e. one cannot just choose arbitrary subspaces to get subrepresentations.

Example 3.6. If we consider the example of Q being 

then a representation of Q is simply a vector space V_1 together with a linear map $\varphi_\alpha : V_1 \to V_1$. Then a subspace $W_1 \subset V_1$ is a subrepresentation precisely if $\varphi_\alpha(W_1) \subseteq W_1$.

We say that the representation M is *simple* if the only subrepresentations of M are M itself and 0. Here is a construction of some simple representations. For each $i \in Q_0$, we let $E(i) = \{(E(i)_j, \varphi_\alpha)\}$ denote the representation where $E(i)_i = \mathbb{C}$ and $E(i)_j = 0$ for $j \neq i$. Moreover, $\varphi_\alpha = 0$ for all α. We leave it to the reader to check that $E(i)$ is simple.

Given two representations $M = \{(V_i, \varphi_\alpha)\}$ and $N = \{(U_i, \psi_\alpha)\}$ we can construct a new representation $M \oplus N$, the *direct sum* of M and N, as follows. At each vertex i, we simply take the direct sum of vector spaces $V_i \oplus U_i$. Then, we define the new map $\eta_\alpha = (\varphi_\alpha, \psi_\alpha)$ for each $\alpha \in Q_1$ i.e.

$$\eta_\alpha = \begin{pmatrix} \varphi_\alpha & 0 \\ 0 & \psi_\alpha \end{pmatrix} : V_{t(\alpha)} \oplus U_{t(\alpha)} \longrightarrow V_{h(\alpha)} \oplus U_{h(\alpha)}.$$

Notice that M and N are naturally subrepresentions of the direct sum $M \oplus N$.

Of course, to better understand representations of a given quiver, we want to be able to compare them. The standard way of doing this is to consider *homomorphisms* between representations. This basic notion is very important in the theory. See the exercises for more details.

3.4 The path algebra

There is a natural algebra we can associate to each quiver Q. Recall that an algebra A is a $\mathbb{C}$-vector space together with a bilinear multiplication map $A \times A \to A$ that makes it into a ring with identity. First we define a *path* in Q to be a tuple of arrows $\alpha_k \alpha_{k-1} \cdots \alpha_1$ such that $h(\alpha_i) = t(\alpha_{i+1})$ for $i = 1, \ldots, k-1$. The length of the path is k. We think of each vertex e_i as being a path of length zero i.e. the path that does nothing. Let $\mathbb{C}Q$ be the complex vector space with basis given by all paths in Q. To define a multiplication on $\mathbb{C}Q$, it suffices to do so for paths. Then,

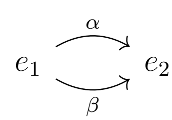
$$(\beta_l \cdots \beta_1) \circ (\alpha_k \cdots \alpha_1) = \begin{cases} \beta_l \cdots \beta_1 \alpha_k \cdots \alpha_1 & \text{if } h(\alpha_k) = t(\beta_1) \\ 0 & \text{otherwise.} \end{cases}$$

That is, multiplication is simply the concatenation of paths where-ever that makes sense. For the paths e_i of length zero, we have

$$(\alpha_k \cdots \alpha_1) \circ e_i = \begin{cases} \alpha_k \cdots \alpha_1 & \text{if } t(\alpha_1) = i \\ 0 & \text{otherwise.} \end{cases}$$

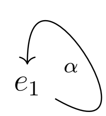
Example 3.7. The path algebra $\mathbb{C}Q$ of

Q :



has basis $\{e_1, e_2, \alpha, \beta\}$ and multiplication is given by

$$e_2 \cdot \alpha = \alpha \cdot e_1 = \alpha, \quad e_2 \cdot \beta = \beta \cdot e_1 = \beta, \quad \alpha \cdot \beta = \beta \cdot \alpha = 0, \quad \ldots$$

But notice that the path algebra $\mathbb{C}Q'$, where Q' : 

has basis $\{e_1, \alpha, \alpha^2, \alpha^3, \ldots\}$ and multiplication

$$e_1 \cdot \alpha^n = \alpha^n \cdot e_1 = \alpha^n, \quad \alpha^n \cdot \alpha^m = \alpha^{n+m}.$$

In particular, $\mathbb{C}Q'$ is infinite dimensional and commutative.

In group theory, it is often very advantageous to study the way that groups can *act* on other objects such as sets, vector spaces, manifolds... Similarly, to better understand an algebra, it is very advantageous to study *actions* of the algebra on vector spaces. This is the field of *representation theory*. At its heart is the notion of a module over the algebra A.

Definition 3.8. An A-*module* M is a vector space equipped with a bilinear map $A \times M \to M$, $(a, m) \mapsto a \cdot m$ such that

- $1 \cdot m = m$ for all $m \in M$.
- $a \cdot (b \cdot m) = (a \cdot b) \cdot m$ for all $a, b \in A$ and $m \in M$.

We will be interested, in particular, in the category of modules for the path algebra $\mathbb{C}Q$.

Exercise 3.9. Notice that the definition of a module uses explicitly the identity element 1 of the algebra A. However, we've made no mention of an identity for the path algebra. Check that

$$1 := \sum_{i \in Q_0} e_i$$

is the identity in $\mathbb{C}Q$.

Example 3.10. If Q is the quiver $e_1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} e_2$ then we could take $M = \mathbb{C}f_1 \oplus \mathbb{C}f_2$ with $e_i \cdot f_j = \delta_{i,j} f_j$ and

$$\alpha \cdot f_1 = 4f_2, \quad \alpha \cdot f_2 = 0, \quad \beta \cdot f_1 = 12f_2, \quad , \beta \cdot f_2 = 0,$$

or $N = \mathbb{C}^7$, where $e_1 \cdot v = v$ and $e_2 \cdot v = \alpha \cdot v = \beta \cdot v = 0$ for all $v \in \mathbb{C}^7$.

3.5 Representations vs. $\mathbb{C}Q$-modules

We have seen that we can associated to a quiver its category of representations and also the category of modules over the path algebra. It turns out that these are equivalent notions (formally, one says that the categories are equivalent). To show that they are equivalent, we explain how to go from a representation to a module for the path algebra and back.

Let $M = \{(V_i, \varphi_\alpha) \mid e_i \in Q_0,\ \alpha \in Q_1\}$ be a representation of Q. We construct out of M a module $F(M)$ for the path algebra $\mathbb{C}Q$. As a vector space,

$$F(M) := \bigoplus_{e_i \in Q_0} V_i,$$

is just the direct sum of the spaces V_i. Since every path in $\mathbb{C}Q$ is the composition of a series of arrows, we just have to say how a) the trivial paths e_i act on $F(M)$ b) how each of the arrows $\alpha \in Q_1$ acts on $F(M)$. Firstly, for each i we have a projection map $p_i : F(M) \to V_i$ and an inclusion map $q_i : V_i \hookrightarrow F(M)$. We define

$$e_i = q_i \circ p_i \quad \text{i.e.} \quad e_i \cdot v = q_i \circ p_i(v).$$

Next, associated to $\alpha \in Q_1$ is the linear map $\varphi_\alpha : V_{t(\alpha)} \to V_{h(\alpha)}$. We use a similar trick to extend this to a map $F(M) \to F(M)$ by setting

$$\alpha = q_{h(\alpha)} \circ \varphi_\alpha \circ p_{t(\alpha)} : F(M) \xrightarrow{q_{h(\alpha)}} V_{t(\alpha)} \xrightarrow{\varphi_\alpha} V_{h(\alpha)} \xrightarrow{p_{t(\alpha)}} F(M),$$

i.e. $\alpha \cdot v = q_{h(\alpha)} \circ \varphi_\alpha \circ p_{t(\alpha)}(v)$.

Now we want to go in the opposite direction. Given a module N for the path algebra $\mathbb{C}Q$, we want to define a representation $G(N) = \{(U_i, \psi_\alpha) \mid e_i \in Q_0,\ \alpha \in Q_1\}$. Notice first that the identity element 1 in $\mathbb{C}Q$ is the sum $\sum_{i \in Q_0} e_i$. Also, the rules for mutiplication in $\mathbb{C}Q$ show that $e_i \cdot e_j = \delta_{i,j} e_i$ (we say that $\{e_i \in Q_0\}$ form a complete set of *orthogonal idempotents* in $\mathbb{C}Q$). This implies that

$$N = \bigoplus_{i \in Q_0} e_i \cdot N.$$

We set $U_i = e_i \cdot N$. We just need to define a linear map $\psi_\alpha : e_{t(\alpha)} N \to e_{h(\alpha)} N$ for each $\alpha \in Q_1$. For this, notice that

Theorem 3.11. *The maps F and G define inverse equivalences between the category of all representations of Q and the category of all $\mathbb{C}Q$-modules.*

Proof. We have done all the hard work already in defining the maps F and G. It is straightforward to check that $G \circ F(M) = M$ for any representation M and $F \circ G(N) = N$ for any $\mathbb{C}Q$-module N. □

Example 3.12. Let's take Q to be the quiver $e_1 \xrightarrow{\alpha} e_2 \xleftarrow[\beta]{} e_3$ and the representation N : $\mathbb{C} \xrightarrow{3} \mathbb{C} \xleftarrow[6]{} \mathbb{C}$. Then $\mathbb{C}Q$ has basis $\{e_1, e_2, e_3, \alpha, \beta\}$ and $F(N)$ is the three-dimensional module with basis f_1, f_2, f_3 such that

$$e_i \cdot f_j = \delta_{i,j}, \quad \alpha \cdot f_1 = 3f_2, \quad \beta \cdot f_3 = 6f_2,$$

and $\alpha \cdot f_i = \beta \cdot f_j = 0$ otherwise. Similarly, if we take the $\mathbb{C}Q$-module M in example 3.10 then $G(M)$ is the representation of Q given by

$$\mathbb{C} \underset{7}{\overset{4}{\rightrightarrows}} \mathbb{C}.$$

Remark 3.13. In modern terms, the above theorem should really be phrased in the language of category theory. The problem with this technical language is that it hides completely the very simple idea behind Theorem 3.11. None the less, I'll give the extra details for those of you of a more abstract nature. Firstly, we are really considering the *categories* $\mathrm{Rep}(Q)$ of representations of Q and $\mathbb{C}Q\text{-}\mathsf{Mod}$ of left $\mathbb{C}Q$-modules. This is not only the data of the set (or rather *class*) $\mathrm{Ob}(\mathrm{Rep}(Q))$ of all representations, but also for each pair of representations $M, N \in \mathrm{Ob}(\mathrm{Rep}(Q))$, the vector space $\mathrm{Hom}_Q(M,N)$ of all morphisms $M \to N$. Similarly, the objects $\mathrm{Ob}(\mathbb{C}Q\text{-}\mathsf{Mod})$ of $\mathbb{C}Q\text{-}\mathsf{Mod}$ are the modules and $\mathrm{Hom}_{\mathbb{C}Q}(M,N)$ is the vector space of morphisms of $\mathbb{C}Q$-modules. Then F and G are *functors* between these two categories i.e. they are defined not only on the objects but also on the hom-spaces of these categories,

$$F_{M,N} : \mathrm{Hom}_Q(M,N) \to \mathrm{Hom}_{\mathbb{C}Q}(F(M), F(N)), \quad f \mapsto F(f),$$

in a way that is compatible with compositions of morphisms. Finally, Theorem 3.11 says that F and G are quasi-inverse equivalences of categories.

3.6 Remarks

There is a huge body of literature on quiver representations, simply type "Quiver representations" into google and see what you get. Here are some references [5], [15], [18], [8], [7], that I found useful when learning about quivers.

4 Gabriel's Theorem

In the previous lecture we were introduced to quivers and their representations. In particular, we saw that the "smallest" possible representations are the simple ones and, given any two representations, we can make a new representation by taking their direct sum. Flipping this on its head, we can ask if a representation can be decomposed into a direct sum of two smaller representations. We say that a representation is *indecomposable* if it cannot be written as the direct sum of two subrepresentations. In this final lecture, we'll consider the question

Q. Which quivers have only finitely many indecomposable representations?

The proof of Gabriel's Theorem is long and difficult, so to properly understand it you'll need to read several different presentations. In the remarks at the end of the section I've listed some sources that contain a complete proof of the theorem.

4.1 Indecomposable representations

In order to study the representations of a quiver, the first thing to do is try and break a representation up into a direct sum of simpler representations - we say that M *decomposes* as a direct sum $M_1 \oplus M_2$. We've already seen that many representations occur as the direct sum of smaller representations. Repeating, we get smaller and smaller representations. Of course this process must eventually stop. That is, eventually we get to a representation M that cannot be written as the direct sum $M_1 \oplus M_2$ of two subrepresentations.

Definition 4.1. The representation M is said to be *indecomposable* if it cannot be decomposed into a direct sum of two proper subrepresentations.

Notice that we can always decompose $M = M \oplus 0$, but such a decomposition doesn't help us understand M better, so we ignore these trivial decompositions. Clearly, if we want to understand, or *classify* all representations of Q, it suffices to classify the indecomposable representations. In general this turns out to be a very difficult (in fact, impossible; see section 4.5) task. It is because of this that it's so important to know which quivers only have finitely many indecomposables.

4.2 Gabriel's Theorem

Here is the statement of Gabriel's Theorem. The proof is rather long, but it will bring together ideas from representation theory, geometry, reflection groups and combinatorics in a very clever way.

Theorem 4.2 (Gabriel). *1. The quiver Q admits finitely many indecomposables (up to isomorphism) if and only if the underlying graph is of type* A, D *or* E.

2. In the case where Q has underlying graph a positive definite Euler graph, $M \mapsto \underline{\dim}\ M$ defines a bijection between the isomorphism classes of indecomposable representations and the positive roots R^+ of the corresponding root system.

Part (2) means that each indecomposable M of Q is uniquely defined, up to isomorphism, by its dimension vector *and* given any $\mathbf{v} \in R^+$ one can construct an indecomposable representation M of Q such that $\underline{\dim}\ M = \mathbf{v}$. This latter statement is the really difficult part of Gabriel's theorem.

Remark 4.3. There is a much more general result, called *Kac's Theorem*; see [13]. This theorem gives a precise description of the set of dimension vectors of the indecomposable representations for *any* quiver Q. Moreover, Kac defined "real" and "imaginary" dimension vectors, and showed that there is only one indecomposable representation of Q with dimension vector $\mathbf{v}$ if $\mathbf{v}$ is a real vector and *infinitely many* indecomposable representations of Q with dimension vector $\mathbf{v}$ if $\mathbf{v}$ is an imaginary vector. In this more general setting, Gabriel's Theorem is saying that every root is real if the underlying graph of Q is positive definite.

4.3 The proof part I: orbits

In this section we prove the easier direction of Gabriel's Theorem: if Q has only finitely many indecomposable representations then the underlying graph must be of type A, D or E. Firstly, we introduce the space $\mathrm{Rep}(Q, \mathbf{v})$ of all representations with dimension vector $\mathbf{v}$. If we have vector spaces of dimension m and n respectively, then the space $\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^n)$ of all linear maps from $\mathbb{C}^m$ to $\mathbb{C}^n$ has dimension $m \times n$. Now, to define a representation M of Q with dimension vector $\mathbf{v} = (v_1, \dots, v_k)$ we choose, for each $\alpha \in Q_1$, an element $\varphi_\alpha \in \mathrm{Hom}(\mathbb{C}^{v_{t(\alpha)}}, \mathbb{C}^{v_{h(\alpha)}})$. Thus, M is a point in

$$\mathrm{Rep}(Q, \mathbf{v}) = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}(\mathbb{C}^{v_{t(\alpha)}}, \mathbb{C}^{v_{h(\alpha)}}).$$

The space $\mathrm{Rep}(Q, \mathbf{v})$ is a vector space of dimension

$$\sum_{\alpha \in Q_1} v_{t(\alpha)} v_{h(\alpha)}.$$

What does it mean for two representations $M, N \in \mathrm{Rep}(Q, \mathbf{v})$ to be isomorphic? Well, if $M = \{(\mathbb{C}^{v_i}, \varphi_\alpha)\}$ and $N = \{(\mathbb{C}^{v_i}, \psi_\alpha)\}$ it means that there are invertible linear maps $f_i \in GL(\mathbb{C}^{v_i})$ for each $i \in Q_0$ such that the diagrams

$$\begin{array}{ccc} \mathbb{C}^{v_{t(\alpha)}} & \xrightarrow{\varphi_\alpha} & \mathbb{C}^{v_{h(\alpha)}} \\ \Big\downarrow f_{t(\alpha)} & & \Big\downarrow f_{h(\alpha)} \\ \mathbb{C}^{v_{t(\alpha)}} & \xrightarrow{\psi_\alpha} & \mathbb{C}^{v_{h(\alpha)}} \end{array}$$

commute for all $\alpha \in Q_1$. That is, $\psi_\alpha = f_{h(\alpha)} \circ \varphi_\alpha \circ f_{t(\alpha)}^{-1}$. Define an action of the group $G(\mathbf{v}) := \prod_{i \in Q_0} GL(\mathbb{C}^{v_i})$ on the vector space $\mathrm{Rep}(Q, \mathbf{v})$ by

$$\mathbf{f} \cdot (\varphi_\alpha) = (f_{h(\alpha)} \circ \varphi_\alpha \circ f_{t(\alpha)}^{-1}), \quad \forall\, \mathbf{f} \in G(\mathbf{v}).$$

Then we have shown that:

Lemma 4.4. *The two representations $M, N \in \mathrm{Rep}(Q, \mathbf{v})$ are isomorphic if and only if N is in the $G(\mathbf{v})$-orbit of M.*

Now we'll need some basic facts from algebraic geometry (this is the study of spaces defined by systems of polynomial equations). These spaces are called *algebraic varieties* e.g. the elliptic curve $C = \{(x, y) \mid y^2 = x^3 + x^2\} \subset \mathbb{C}^2$ is an algebraic variety. Just as for a vector space, these spaces also have "dimension"; the dimension of C is one, as you might expect. Now the key fact we'll use is that the $G(\mathbf{v})$-orbits in $\mathrm{Rep}(Q, \mathbf{v})$ are not just sets, they are also algebraic varieties! Therefore, we can compute the dimension of an orbits $G(\mathbf{v}) \cdot M := \{\mathbf{f} \cdot (\varphi_\alpha) \mid \mathbf{f} \in G(\mathbf{v})\}$. Moreover, there's a geometric analogue of the classical orbit-stabilizer theorem; see [5, Proposition 2.1.6] for a proof.

Proposition 4.5. *Let $M \in \mathrm{Rep}(Q, \mathbf{v})$. Then,*

$$\dim G(\mathbf{v}) \cdot M = \dim G(\mathbf{v}) - \dim \mathrm{Stab}_{G(\mathbf{v})}(M).$$

What is the dimension of $G(\mathbf{v})$? Well, the set (in fact, variety) of all invertible linear maps $GL(\mathbb{C}^n)$ is an open subset of all linear maps $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$. This implies that $\dim GL(\mathbb{C}^n) = \dim \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) = n^2$. Therefore,

$$\dim G(\mathbf{v}) = \dim \prod_{i \in Q_0} GL(\mathbb{C}^{v_i}) = \sum_{i \in Q_0} \dim GL(\mathbb{C}^{v_i}) = \sum_{i \in Q_0} v_i^2.$$

Define the *Ringel form* $\langle -, - \rangle$ on $\mathbb{Z}^{Q_0}$ by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i \in Q_0} v_i w_i - \sum_{\alpha \in Q_1} v_{t(\alpha)} w_{h(\alpha)}. \tag{7}$$

Eventually, we want to arrive at a positive definite symmetric form, just as in the classification of finite reflection groups. However, the Ringel form $\langle -, - \rangle$ is not symmetric. Therefore, the *Euler form* is defined to be the symmetrization of the Ringel form,

$$\begin{aligned}(\mathbf{v}, \mathbf{w})_E &:= \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle \\ &= 2 \sum_{i \in Q_0} v_i w_i - \sum_{\alpha \in Q_1} v_{t(\alpha)} w_{h(\alpha)} + w_{t(\alpha)} v_{h(\alpha)}.\end{aligned}$$

It does not depend on the orientation of Q i.e. it only depends on the underlying graph. But it is important to note that this form we have just associated to the underling graph of Q is *not* the Coxeter form! To distinguish between the two different symmetric forms, we will refer to a graph as an *Euler graph* when we mean that the associated form is the Euler form. Similarly, a graph Γ will be said to be a positive (semi-)definite if the associated Euler form is positive (semi-)definite. Why is the Euler form useful? Let's compute:

$$\begin{aligned}\frac{1}{2}(\mathbf{v}, \mathbf{v})_E &= \sum_{i \in Q_0} v_i^2 - \sum_{\alpha \in Q_1} v_{t(\alpha)} v_{h(\alpha)}, && (8)\\ &= \dim G(\mathbf{v}) - \dim \operatorname{Rep}(Q, \mathbf{v}) && (9)\\ &= \dim G(\mathbf{v}) \cdot M + \dim \operatorname{Stab}_{G(\mathbf{v})}(M) - \dim \operatorname{Rep}(Q, \mathbf{v}), && (10)\end{aligned}$$

where we have used Proposition 4.5 in the last step.

Lemma 4.6. *If Q has only finitely many indecomposable representations then $(\mathbf{v}, \mathbf{v})_E \geq 2$ for all $\mathbf{v} \in \mathbb{N}^{Q_0}$.*

Proof. If there are only finitely many indecomposable representations of Q, then for a fixed dimension vector $\mathbf{v} \in \mathbb{N}^{Q_0}$ there are only finitely many representation of Q up to isomorphism with dimension vector $\mathbf{v}$. Geometrically, this means that there are only finitely many $G(\mathbf{v})$-orbits in $\operatorname{Rep}(Q, \mathbf{v})$. Since the dimension of the union of *finitely many* algebraic varieties is the maximum of the dimension of the varieties, there must be an open orbit in $\operatorname{Rep}(Q, \mathbf{v})$ i.e. an orbit $G(\mathbf{v}) \cdot M$ such that $\dim G(\mathbf{v}) \cdot M = \dim \operatorname{Rep}(Q, \mathbf{v})$. For this orbit, equation (10) implies that $\frac{1}{2}(\mathbf{v}, \mathbf{v})_E = \dim \operatorname{Stab}_{G(\mathbf{v})}(M)$. Thus, we just need to show that $\dim \operatorname{Stab}_{G(\mathbf{v})}(M) \geq 1$. For $\lambda \in \mathbb{C}^\times$, consider the element $\boldsymbol{\lambda} = (\lambda \operatorname{Id}_{\mathbb{C}^{v_i}}) \in G(\mathbf{v})$. Then

$$\boldsymbol{\lambda} \cdot (\varphi_\alpha) = (\lambda^{-1}\varphi_\alpha\lambda^{-1}) = (\varphi_\alpha)$$

i.e. there is a diagonal copy of $\mathbb{C}^\times$ in $\operatorname{Stab}_{G(\mathbf{v})}(M)$ for any representation M. Since $\dim \mathbb{C}^\times = 1$, this implies that $\dim \operatorname{Stab}_{G(\mathbf{v})}(M) \geq 1$, as required. □

Just as in the classification of finite reflection groups, we are aiming to reduce the classification problem of Gabriel's Theorem to one of classifying symmetric bilinear forms constructed from finite graphs.

Proposition 4.7. *Let Q be a quiver with only finitely many indecomposable representations. Then the Euler form $(-,-)_E$ is positive definite.*

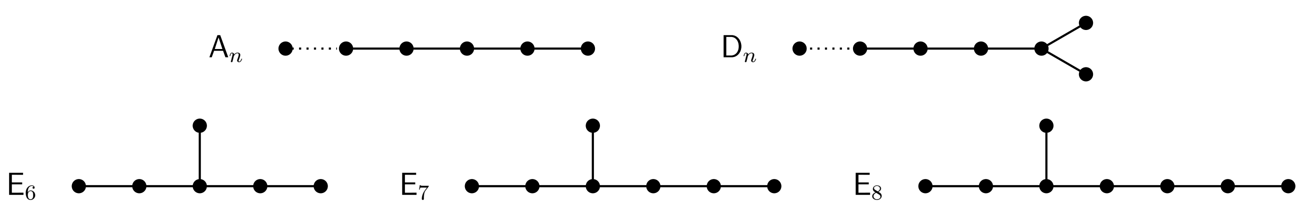
Proof. We need to show that $(\mathbf{v},\mathbf{v})_E > 0$ for all non-zero $\mathbf{w} \in \mathbb{R}^n$. If there exists a non-zero vector $\mathbf{w} \in \mathbb{R}^n$ such that $(\mathbf{v},\mathbf{v})_E \leq 0$ then there exists $\mathbf{w} \in \mathbb{Q}^{Q_0}$ such that $(\mathbf{v},\mathbf{v})_E \leq 0$. Rescaling, we may assume that $\mathbf{w} \in \mathbb{Z}^{Q_0}$. Finally, notice from the explicit formula (7) for $(\mathbf{v},\mathbf{v})_E$ that if $|\mathbf{w}| := (|w_1|, |w_2|, \dots,) \in \mathbb{N}^{Q_0}$ then

$$(|\mathbf{w}|, |\mathbf{w}|)_E \leq (\mathbf{v},\mathbf{v})_E \leq 0.$$

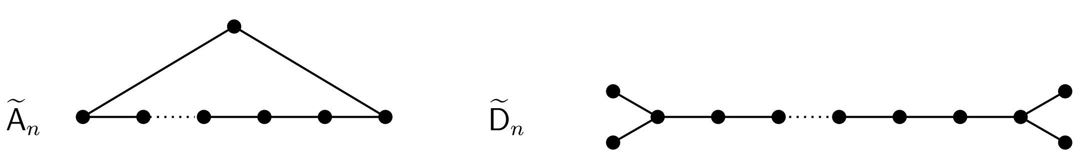
Thus, we have shown that if $(-,-)_E$ is not positive definite then there exists a non-zero vector $\mathbf{w} \in \mathbb{N}^{Q_0}$ such that $(\mathbf{v},\mathbf{v})_E \leq 0$. We deduce from Lemma 4.6 that $(-,-)_E$ is positive definite. □

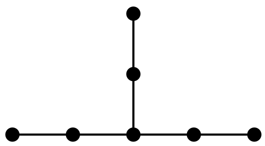
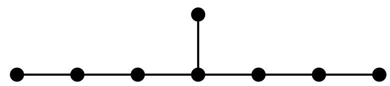
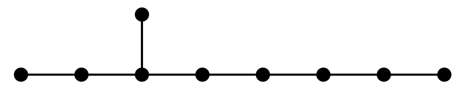
The proof of the following proposition is a direct, inductive, calculation.

Proposition 4.8. *The following Euler graphs are positive definite.*



The following Euler graphs are positive semi-definite.



$\widetilde{\mathsf{E}}_6$  $\widetilde{\mathsf{E}}_7$  $\widetilde{\mathsf{E}}_8$ 

Now we can state the classification theorem for Euler graphs that implies the easy direction in Gabriel's Theorem.

Theorem 4.9. *If Γ is a finite connected graph whose Euler form is positive (semi-)definite, the Γ appears on the list given in Proposition 4.8.*

Proof. As you can see from exercise 4.11, it is possible to deduce the theorem from Theorem 2.21. Here we will give a direct proof.

Just as in the proof of Theorem 2.21, the key result is Corollary 2.20, though the analysis is much simpler in this case. We will concentrate on classifying the positive definite graphs (we do not actually need Corollary 2.20 for this), the argument for positive semi-definite graphs is similar. So we assume that Γ is positive definite. This means that no subgraph of Γ can be positive semi-definite. Recall that the *valence* of a vertex is the number of edges leaving the vertex. Hence,

(a) Γ contains no cycles ($\widetilde{\mathsf{A}}_n$),

(b) Each vertex of Γ has valence at most 3 ($\widetilde{\mathsf{D}}_4$),

(c) There is at most one vertex with valence 3 ($\widetilde{\mathsf{D}}_n$),

(d) If every vertex has valence at most 2 then Γ is type A,

(e) We assume that there is one vertex of valence 3. Then $\Gamma = T(p, q, r)$ is a spoke with arms of length p, q, r (so that Γ has $p + q + r$ edges in total).

(f) At least one of p, q, r is equal to 1 ($\widetilde{\mathsf{E}}_6$). So without loss of generality, $p = 1$.

(g) $q \leq 2$ ($\widetilde{\mathsf{E}}_7$). If $q = 1$ then Γ is of type D. Thus, we assume $q = 2$.

(h) $r \leq 4$ ($\widetilde{\mathsf{E}}_8$) and hence Γ is of type E.

□

Exercise 4.10. Using Corollary 2.20 and the arguments of the proof of Theorem 4.9, show that the positive semi-definite Euler forms listed in Proposition 4.8 constitute all such forms.

Exercise 4.11. You'll notice that the positive definite Euler graphs are precisely the positive definite Coxeter graphs that are *simply laced* i.e. have at most one edge between any two vertices. Let $(-,-)_C$, resp. $(-,-)_E$, be the Coxeter form, resp. the Euler form, associated to a graph Γ.

1. Show that if Γ is simply laced then $(-,-)_E = 2(-,-)_C$.

2. If Γ is not simply laced, show that there is no $\lambda \in \mathbb{R}$ such that $(-,-)_E = \lambda(-,-)_C$.

3. Show that the symmetric matrix
$$\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$$
corresponding to the Euler graph $\bullet\!\overset{m}{\text{———}}\!\bullet$ is positive definite if and only if $m = 1$. When is it positive semi-definite?

4. By considering the subgraphs $\bullet\!\overset{m}{\text{———}}\!\bullet$ with $m > 1$ of Γ, show that a non-simply laced Euler graph is not positive definite.

5. Deduce Theorem 4.9 from Theorem 2.21.

4.4 The proof part II: reflection functors

In order to relate the representation theory of Q to the corresponding root system, we will recall the proof, in terms of reflection functors, given by Bernstein, Gelfand and Ponemarev. The presentation we give here is based on unpublished lecture notes by H. Derksen.

Since we have defined the Euler form $(-,-)_E$ on $\mathbb{Z}^{Q_0}$, we can define, for each *loop free vertex* $i \in Q_0$, a reflection s_i by
$$s_i(\mathbf{v}) = \mathbf{v} - (\mathbf{v}, e_i)e_i.$$
Then, if $\mathbf{w} = s_i(\mathbf{v})$, we have $\mathbf{w}_j = \mathbf{v}_j$ for $j \neq i$ and
$$\mathbf{w}_i = -\mathbf{v}_i + \sum_{h(\alpha)=i} \mathbf{v}_{t(\alpha)} + \sum_{t(\alpha)=i} \mathbf{v}_{h(\alpha)}.$$
In this way we get a reflection group $W(Q)$ acting on $\mathbb{Z}^{Q_0}$.

Remark 4.12. In general, $W(Q)$ will be an infinite group. It will be the same as the Coxeter group associated to the underlying graph of Q, as defined in Theorem 2.8, if and only if Q is simply laced. One can prove this using the results of exercise 4.11.

We call a vertex $i \in Q_0$ a *sink* if there are no arrows emanating from i i.e. there is no $\alpha \in Q_1$ such that $t(\alpha) = i$. Similarly, i is said to be a *source* if there are no arrows ending in i i.e. there is no $\alpha \in Q_1$ such that $h(\alpha) = i$. For each sink or source i of Q, we will define a *reflection functor* at i that, given a representation M of Q, will produce a new representation whose dimension vector is $s_i(\underline{\dim}\ M)$. This will allow us to construct all indecomposable representations of Q starting from the simple representations $E(i)$.

First, we assume that i is a sink. We define a functor $S_i^+ : \mathrm{Rep}(Q) \to \mathrm{Rep}(s_iQ)$, where s_iQ is the quiver obtained by reversing all the arrows heading into i (so that i becomes a source in s_iQ). Let $M = \{(V_j, \varphi_\alpha)\}$ be a representation of Q. Then $S_i^+(M) = \{(U_j, \psi_\alpha)\}$, where $U_j = V_j$ for all $j \neq i$, and if $\alpha \in Q_1$ with $h(\alpha), t(\alpha) \neq i$, then $\psi_{s_i(\alpha)} = \varphi_\alpha$. At i, we define U_i to be the kernel of the map

$$\bigoplus_{h(\alpha)=i} V_{t(\alpha)} \to V_i, \quad (v_{t(\alpha)}) \mapsto \sum_{h(\alpha)=i} \varphi_\alpha(v_{t(\alpha)}).$$

If $h(\alpha) = i$ then $t(s_i(\alpha)) = i$ in s_iQ and we define

$$\psi_{s_i(\alpha)} : U_i \to U_{h(s_i(\alpha))} = V_{t(\alpha)}$$

to be the restriction to U_i of the projection map $\bigoplus_{h(\alpha)=i} V_{t(\alpha)}$. The functor S_i^+ acts on morphisms between representations in the obvious way.

Next we consider the case where i is a source. We define a functor $S_i^- : \mathrm{Rep}(Q) \to \mathrm{Rep}(s_iQ)$. Again, let $M = \{(V_j, \varphi_\alpha)\}$ be a representation of Q. Then $S_i^-(M) = \{(U_j, \psi_\alpha)\}$, where $U_j = V_j$ for all $j \neq i$, and if $\alpha \in Q_1$ with $h(\alpha), t(\alpha) \neq i$, then $\psi_{s_i(\alpha)} = \varphi_\alpha$. At i, we define U_i to be the cokernel of the map

$$V_i \to \bigoplus_{t(\alpha)=i} V_{h(\alpha)}, \quad v_i \mapsto (\varphi_\alpha(v_i)).$$

If $t(\alpha) = i$ then $h(s_i(\alpha)) = i$ in s_iQ and we define

$$\psi_{s_i(\alpha)} : U_{t(s_i(\alpha))} = V_{h(\alpha)} \to U_i$$

to be the composition of the embedding $V_{h(\alpha)} \hookrightarrow \bigoplus_{t(\beta)=i} V_{h(\beta)}$ followed by the quotient map $\bigoplus_{t(\beta)=i} V_{h(\beta)} \to U_i$. Just as for sinks, it is straight-forward to define the functor S_i^- on morphisms between representations.

Remark 4.13. What we have done is take an algebraic structure, in this case a reflection s_i, and "lift" it to a functor S_i^+ between categories. Thus, the categories $\text{Rep}(Q)$ and $\text{Rep}(s_iQ)$ are playing the role of $\mathbb{Z}^{Q_0}$. This notion of lifting algebraic structures, in particular representations, to categories and functors between them is called *categorification*. It is a relative new concept, but one that has turned out to be incredibly powerful. As such, a great deal of research is currently devoted to "categorifying" representations. For more on this fascinating subject, see [14] and [16].

The following theorem is the key result needed to complete the proof of Gabriel's Theorem. The proof is long and involved, so it won't be given here. For the details see the references given at the end of the section.

Theorem 4.14 (Bernstien-Gelfand-Ponomarev)**.** *Let $i \in Q_0$ be a sink and M an indecomposable representation of Q. Either,*

1. *$M = E(i)$ and $S_i^+(E(i)) = 0$; or*
2. *$S_i^+(M) \neq 0$ is indecomposable $S_i^- S_i^+(M) \simeq M$ and $\underline{\dim}\ S_i^+(M) = s_i(\underline{\dim}\ M)$.*

Let $i \in Q_0$ be a source and M an indecomposable representation of Q. Either,

1. *$M = E(i)$ and $S_i^-(E(i)) = 0$; or*
2. *$S_i^-(M) \neq 0$ is indecomposable $S_i^+ S_i^-(M) \simeq M$ and $\underline{\dim}\ S_i^-(M) = s_i(\underline{\dim}\ M)$.*

From now on, we assume that the underlying graph of Q contains no cycles (this is no big restriction since we ultimately only want to consider the quivers of type A, D or E). Then we can relabel the vertices $Q_0 = \{1, \dots, n\}$ of Q so that $t(\alpha) < h(\alpha)$ for all $\alpha \in Q_1$. We define the *Coxeter element* in $W(Q)$ as

$$c = s_1 \cdots s_k.$$

Its inverse is $c^{-1} = s_k \cdots s_1$. A moment's thought shows that $cQ = Q$ since each arrow is reversed exactly twice. This means that we can lift c and c^{-1} to functors

$$C^+ := S_k^+ \circ \cdots \circ S_1^+ :\ \text{Rep}(Q) \to \text{Rep}(Q)$$

and

$$C^- := S_1^- \circ \cdots \circ S_k^- :\ \text{Rep}(Q) \to \text{Rep}(Q).$$

Theorem 4.14 now implies

Corollary 4.15. *Let M be an indecomposable representation of Q.*

1. *Either $C^+(M) = 0$ or $C^+(M)$ is an indecomposable representation such that $\underline{\dim}\ C^+(M) = c(\underline{\dim}\ M)$.*

2. *Either $C^-(M) = 0$ or $C^-(M)$ is an indecomposable representation such that $\underline{\dim}\ C^-(M) = c^{-1}(\underline{\dim}\ M)$.*

From now until the end of the proof of Gabriel's Theorem, we assume that Q is of type A, D or E. Then, as noted in exercise 4.11, the two symmetric forms $(-,-)_E$ and $(-,)_C$ agree and we can think of $\mathbb{Z}^{Q_0}$ as a subset of $\mathbb{R}^n$ by sending the dimension vectors e_i to the basis of $\mathbb{R}^n$ given by the simple roots $\Delta = \{e_1, \dots, e_n\}$. Since the underlying graph of Q is of crystallographic type (see section 2.4), we may assume that the root system $\Delta \subset R$ is of crystallographic type. Notice that we have a preferred set of positive roots $R^+ \subset \mathbb{N}^{Q_0}$ since we have specified Δ. The reflection group $W(Q)$ acts on R and Theorem 2.7 implies that $W(Q) \cdot \Delta = R$ i.e each α in R can be written (non-uniquely) as $w(e_i)$ for some $w \in W(Q)$ and $e_i \in \Delta$.

Lemma 4.16. *Let W be a reflection group of type A, D or E and $\alpha \in \mathbb{N}^{Q_0}$ a non-zero dimension vector. Then $c(\alpha) \neq \alpha$ and there exists some $k \gg 0$ such that $c^k(\alpha) \notin \mathbb{N}^{Q_0}$.*

Proof. For the first part of the statement, we require the following identity

$$\langle \alpha, c(\beta) \rangle = -\langle \beta, \alpha \rangle, \quad \forall\ \alpha, \beta \in \mathbb{Z}^{Q_0}. \tag{11}$$

Unfortunately, the proof of this equation uses the notion of projective and injective representations, but if you'd like to see the proof see Lemma 4.4.1 and Lemma 4.4.2 of [15]. Assume that there exists some $\alpha \neq 0$ such that $c(\alpha) = \alpha$. Then equation (11) implies that

$$(\alpha, \alpha) = \langle \alpha, c(\alpha) \rangle + \langle \alpha, \alpha \rangle = -\langle \alpha, \alpha \rangle + \langle \alpha, \alpha \rangle = 0.$$

But we have shown that the Euler form is positive definite for Q. Therefore this implies that $\alpha = 0$; a contradiction.

For the second part, notice that c has finite order since W is finite. Therefore, if the order of c is n say, the root $\alpha + c(\alpha) + \cdots + c^{n-1}(\alpha)$ is invariant under c, hence it is zero. This implies that some $c^k(\alpha)$ does not belong to $\mathbb{N}^{Q_0}$. □

Finally we can proof the hard part of Gabriel's Theorem.

Theorem 4.17. *Let Q be a quiver, whose underlying graph is of type A, D or E. Then there is an indecomposable of dimension α if and only if $\alpha \in R^+$. Moreover, any such indecomposable is unique, up to isomorphism.*

Proof. Let M be an indecomposable representation of Q with dimension vector α. By Lemma 4.16, there exists some $k > 0$ such that $c^k(\alpha) \notin \mathbb{N}^{Q_0}$. This means that $(C^+)^k(M) = 0$. Choose k minimal such that $(C^+)^k(M) = 0$. Then, if $N = (C^+)^{k-1}(M)$, we have $N \neq 0$ and $C^+(N) = 0$. Then Corollary 4.15 implies that there is some i such that $S_i^+ \circ \cdots \circ S_1^+(N) = E(i+1)$ is simple with dimension vector e_{i+1}. It also implies that

$$M = (C^-)^{k-1}(N) = (C^-)^{k-1} \circ S_1^- \circ \cdots \circ S_i^-(E(i+1)).$$

This shows that $\alpha = c^{k-1} s_1 \cdots s_i(e_{i+1})$ is a positive root and M is uniquely defined by its dimension vector.

Finally, we just need to show that if β is an arbitrary positive root, then there exists an indecomposable M with dimension vector β. Again, by Lemma 4.16, we can choose $k > 0$ such that $c^k(\beta) \notin \mathbb{N}^{Q_0}$ but $c^{k-1}(\beta) \in \mathbb{N}^{Q_0}$. Corollary 4.15 implies that $s_i s_{i-1} \cdots s_1(\beta) = e_{i+1}$ for some i. In this case, we can take $M = (C^-)^{k-1} \circ S_1^- \circ \cdots \circ S_i^-(E(i+1))$. Corollary 4.15 ensures that M is an indecomposable with dimension vector β. □

Remark 4.18. For an alternative proof using the geometry of orbit closures, see [7] or [5].

4.5 Finite, tame and wild

What about the positive semi-definite Euler graphs? Do their quiver representations have some particularly interesting properties? The answer is yes, but it order to give you the correct statement, I need to tell you something about tame and wild quivers. We know already that if Q is not of positive definite type then there will be infinitely many indecomposable representations up to isomorphism. Even if we fix a dimension vector $\mathbf{v}$, this will still be the case. However, in many examples one sees that we can still parameterize the indecomposables as an infinite family of modules. For instance, if we consider the quiver $1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$ with dimension vector $(1,1)$ then the indecomposables are in bijection with points on the projective line $\mathbb{P}^1$. We say that Q is *tame* if, for any fixed dimension vector, there are at most finitely many one-parameter families of indecomposable representations of Q. Otherwise Q is said to be *wild*. Kac's Theorem says that Q is tame if and only if the underlying graph is a positive semi-definite Euler graph. In the above example, the underlying graph is of type $\widetilde{A}_2$. For details on this, see [1, Section 4.4].

Example 4.19. The following example is very classical and can be viewed as one of the motivating example behind the definition of a quiver. Fix a vector space $\mathbb{C}^n$ and consider the problem of classifying k-tuples of subspaces $(V_1, \ldots, V_k)$ with $\dim V_i = n_i \leq n$. We don't care about change of basis in $\mathbb{C}^n$, so we think of $(V_1, \ldots, V_k)$ as equivalent to $(U_1, \ldots, U_k)$ if there is some $g \in GL(\mathbb{C}^n)$ such that

$$(U_1, \ldots, U_k) = (gV_1g^{-1}, \ldots, gV_kg^{-1}).$$

We can encode this problem as a problem about quiver representations. Let Q be the spoke quiver with one central vertex 0 and k surrounding vertices $1, \ldots, k$ with an arrow α_i from each outer vertex i to 0. When $k = 4$, we have

$$\begin{array}{ccccc} & & 1 & & \\ & & \downarrow{\scriptstyle\alpha_1} & & \\ 2 & \xrightarrow{\alpha_2} & 0 & \xleftarrow[\alpha_4]{} & 4 \\ & & {\scriptstyle\alpha_3}\uparrow & & \\ & & 3 & & \end{array}$$

If we take the dimension vector $\mathbf{v} = (n, n_1, \ldots, n_k)$, then classifying the subspaces is the same as classifying representations of Q with dimension vector $\mathbf{v}$ such that each φ_{α_i} is injective - a subset of all representations. Then Gabriel's Theorem says that there are only finitely many collections of subspaces when $k \leq 3$. When $k = 4$, we get the Euler graph $\widetilde{D}_4$, which is positive semi-definite by Proposition 4.8. Therefore, for each $\mathbf{v}$ there are finitely many 1-dimensional families of collections of subspaces. When $k > 4$, the classification problem is wild.

4.6 The universality of Coxeter/Euler graphs

Behind much of what we have seen already is a Coxeter/Euler graph of some type or other. Recall that the set of positive definite Coxeter graphs are the ones of type $\mathsf{A}, \mathsf{B}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{G}, \mathsf{H}$ and I. The graphs of type $\mathsf{A}, \mathsf{B}, \mathsf{D}, \mathsf{E}, \mathsf{F}$ and G are the crystallographic Coxeter graphs and the simply laced ones (those of type A, D and E) are precisely the positive definite Euler graphs.

Each of the following classes of objects are classified by Coxeter, or Euler, graphs of some sort. In each case, this is because the classification problem is shown to be equivalent to the classification of Coxeter/Euler graphs of various types.

- The classification of finite reflection groups. We've seen that these are classified by positive definite Coxeter graphs.

- The classification of quivers of finite type. Gabriel's Theorem shows that these are classified by positive definite Euler graphs.

- The classification of simple Lie algebras. This (essentially[2]) reduces to the problem of classifying crystallographic Coxeter graphs; see [11, Chapter 11].

- The classification of rational surface singularities. These are classified by positive definite Euler graphs; see [17].

- The classification, up to conjugation, of finite subgroups of $SU(2)$. As explained in [20], the McKay correspondence shows that these groups are also classified by positive definite Euler graphs.

- The positive definite Euler graphs also classify the "elementary catastrophes" appearing in Thom's Catastrophe theory. See section 9 of [19] for details.

In this list there is no mention of the original objects we started with, the Platonic solids. If you are willing to be a bit flexible with your definitions then they also fit into this classification. The "dihedron" is defined to be the regular p-gon but with an infinitesimal thickening to make it a 3-dimensional solid. It has Schläfli symbol $\{p, 2\}$ for $p \geq 2$. Similarly, the "orange" is the regular solid where each face only has two edge (think of Terry's chocolate orange). It has Schläfli symbol $\{2, q\}$ for $q \geq 3$. In this way we get

- The classification of Platonic solids is given by positive definite Euler graphs.

- There is a bijection between the Platonic solids and the finite subgroups of $SO(3, \mathbb{R})$ (i.e. the finite rotationl groups), up to isomorphism. This is given by sending a solid P to its group $W_0(P)$ of rotational symmetries. Thus, the finite rotation groups are classified by positive definite Euler graphs.

4.7 Where to next?

What we've covered so far is just the tip of a huge, wonderfully complex, iceberg. We've seen how finite reflection groups, root systems, Lie algebras and quiver representations are all related by Coxeter and Euler graphs. These are just some of the many mathematical objects that have these graphs underlying them. Some others are listed in section 4.6 above. Each of these objects is a particular example of a rich story of it's own. For instance rational surface singularities are the some of the most basic examples of singular algebraic varieties in algebraic geometry. This

[2]Strictly speaking they are classified by the corresponding Dynkin diagram, which also encodes the relative length of the roots of the crystallographic root system.

is a vast subject, with a long history, but still the focus of attention for many mathematicians worldwide. A similar statement can be made about Lie theory.

I would just like to mention two key areas that today play a role on the interface between representation theory and geometry and that I think are particularly beautiful generalizations of the above theory. The first of these is the generalized McKay correspondence - the original McKay correspondence relates the representation theory of finite subgroups G of $SU(2)$ with the geometry of the resolution of the corresponding quotient singularity $\mathbb{C}^2/G$. Generalizations of this aim to relate resolutions of the singularities $\mathbb{C}^n/G$, for $G \subset SU(n)$, with the representation theory of G. The most general result in this direction is the generalized McKay correspondence of Bridgeland, King and Reid [4]. A closely related result was shown by Bezrukavnikov and Kaledin [3].

The second is the theory of Nakajima's quiver varieties. These are moduli spaces of representations of quivers, satisfying some additional relations. They can be used for instance to construct resolutions of the quotient singularities $\mathbb{C}^2/G$. But they also appear in the representation theory of certain infinite dimensional Lie algebras called Kac-Moody Lie algebras, are ubiquitous in the construction of categorifications, and have applications in geometry and theoretical physics. For an introduction to quiver varieties, have a look at the lecture notes [9] by Ginzburg.

4.8 Remarks

For a crash course on Gabriel's Theorem and Kac's Theorem, try the very well written article [8]. The extensive notes [15] also contain a detailed proof of Gabriel's Theorem via reflection functors (as well as many other interesting applications of these functors). The original paper [2] by Bernstein-Gelfand-Ponomarev, where they prove Gabriel's Theorem using reflection functors is also very readable - a copy is available at

`http://www.math.tau.ac.il/~bernstei/Publication_list/publication_texts/BGG-CoxeterF-Usp.pdf`

Other notes that contain a proof of Gabriel's Theorem include [5], [18], and [7].

5 Appendix

5.1 The Spectral Theorem for symmetric matrices

The Spectral Theorem for real valued symmetric matrices is a very useful result. We include a proof here.

Theorem 5.1. *Let A be a real valued symmetric matrix.*

1. *The eigenvalues of A are real.*

2. *There exists an orthogonal matrix $g \in \mathrm{O}(n, \mathbb{R})$ such that gAg^T is diagonal.*

Here $\mathrm{O}(n, \mathbb{R}) = \{g \in GL(n, \mathbb{R}) \mid g^{-1} = g^T\}$ is the *real orthogonal Lie group.* You might think that it's not so surprising that a matrix with real coefficients has real eigenvalues. However, most matrices with real coefficients actually have complex eigenvalues. As a simple examples, consider

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 5 \\ -3 & 6 \end{pmatrix}, \quad \begin{pmatrix} 0 & -7 \\ 1 & 12 \end{pmatrix}.$$

The second part of the the spectral theorem is saying that one can always find an orthonormal basis $\{u_1, \dots, u_n\}$ of $\mathbb{R}^n$ i.e. a basis such that $(u_i, u_j) = \delta_{i,j}$, such that each u_i is an eigenvector for A.

Proof of Theorem 5.1. First we note that A is symmetric if and only if $(Au, v) = (u, Av)$ for all $u, v \in \mathbb{R}^n$; it is enough to check this when $u = e_i$ and $v = e_j$ are standard basis elements, then $(Au, v) = a_{j,i}$ and $(u, Av) = a_{i,j}$.

As noted above, (2) is equivalent to showing that there is an orthonormal basis $\{u_1, \dots, u_n\}$ such that each u_i is an eigenvector of A (with eigenvector α_i say). Therefore, we will construct the basis $\{u_1, \dots, u_n\}$ by induction on n and show that each α_i is real. The case $n = 1$ is vacuous since A is real valued. Thus, we may assume by induction that the theorem holds for all symmetric $(n-1) \times (n-1)$ matrices.

In general, a real valued matrix need not have any real eigenvectors. So the first (and in fact hardest) step is to show that A has a real eigenvector with real eigenvalue. To do this, we briefly need to step into the complex world where we are guaranteed the existence of at least one eigenvector. A complex valued matrix B is called *unitary* if $B^\dagger = B$, where $B^\dagger := \left(\overline{B}\right)^T = \overline{(B^T)}$ and $\overline{B}$ means take the complex conjugate of every entry of B. Clearly every real symmetric matrix is unitary when considered as a complex matrix. Let u be an eigenvector for A (over $\mathbb{C}$) with eigenvalue λ. We wish to show that λ is real. If $u^\dagger := \overline{u}^T$ then $u^\dagger \cdot u = \sum_{i=1}^n |u_i|^2 \in \mathbb{R}_{>0}$. Then

$$\overline{(u^\dagger A u)} = \overline{\lambda (u^\dagger \cdot u)} = \overline{\lambda}(u^\dagger u).$$

On the other hand,

$$\begin{aligned}\overline{(u^\dagger A u)} &= u^\dagger A^\dagger u \\ &= u^\dagger A u = \lambda(u^\dagger \cdot u),\end{aligned}$$

where the first equality is true for any matrix and the second is because A is unitary. Hence we deduce that $(\lambda - \overline{\lambda})(u^\dagger \cdot u) = 0$. This means that $\lambda = \overline{\lambda}$ is real. If we write $u = v + \sqrt{-1}w$, where $v, w \in \mathbb{R}^n$, then

$$Au = (Av) + \sqrt{-1}(Aw) = (\lambda v) + \sqrt{-1}(\lambda w).$$

Therefore, both v and w are real eigenvectors for A with eigenvalue λ. Since $u \neq 0$, at least one of them is non-zero. Let u_n be this real eigenvector. Rescaling if necessary, we may assume that $(u_n, u_n) = 1$. Set $\alpha_n := \lambda$.

Now, we'll use the induction hypothesis to find the vectors $u_1, \ldots, u_{n-1}$. Let

$$V = u_n^\perp := \{v \in \mathbb{R}^n \mid (u_n, v) = 0\},$$

an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. Since A is symmetric and $Au_n = \alpha_n u_n$, we have $(Av, u_n) = \alpha_n(v, u_n) = 0$ for all $v \in V$, which means that $A(V) \subset V$. Chose an arbitrary basis for V and write A' for the $(n-1) \times (n-1)$ matrix of A restricted to V. Then

$$A = \begin{pmatrix} A' & 0 \\ 0 & \alpha_n \end{pmatrix}.$$

Moreover, $(A'u, v) = (Au, v) = (u, Av) = (u, A'v)$ for all $u, v \in V$ which means that A' is also a symmetric real matrix. By induction there exists an orthonormal basis $\{u_1, \ldots, u_{n-1}\}$ of V such that $A'u_i = \alpha_i u_i$. Then $\{u_1, \ldots, u_{n-1}, u_n\}$ is the basis of $\mathbb{R}^n$ that we have been looking for (check!). $\square$

5.2 Rotations in $\mathbb{R}^3$

In this section we will prove that

$$\text{The transformation } g \in GL(\mathbb{R}^3) \text{ is a rotation if and only if } g \in SO(3, \mathbb{R}). \tag{12}$$

Here $SO(3, \mathbb{R})$ is the group of all matrices $g \in GL(\mathbb{R}^3)$ such that $\det(g) = 1$ and $gg^T = 1$. A *rotation* of $\mathbb{R}^3$ is a transformation that fixes pointwise a line and acts as a rotation in the plane perpendicular to that line.

We will prove the above in a series of easy claims. We begin with something more general. A complex valued matrix $B \in GL(\mathbb{C}^n)$ belongs to the *unitary group* $U(n)$ if $B^{-1} = B^\dagger$, where $B^\dagger := \left(\overline{B}\right)^T = \overline{(B^T)}$ and $\overline{B}$ means take the complex conjugate of every entry of B. Clearly $O(n, \mathbb{R}) \subset U(n)$.

Lemma 5.2. *Let $A \in U(n)$.*

1. *There exists $g \in U(n)$ such that $gAg^\dagger$ is diagonal.*
2. *The eigenvalues λ_i of A have modulus $|\lambda_i|$ equal to* 1*.*

Proof. We prove both claims by induction on n, just as in the proof of Theorem 5.1. The case $n = 1$ is vacuous.

We extend the usual inner product $(-,-)$ on $\mathbb{R}^n$ to the Hermitian form $\langle -,- \rangle$ on $\mathbb{C}^n$ given explicitly by

$$\langle z, w \rangle = \sum_{i=1}^{n} \overline{z}_i w_i.$$

Then $A \in U(n)$ if and only if $\langle Az, Aw \rangle = \langle z, w \rangle$ for all $z, w \in \mathbb{C}^n$.

Since we are working with complex matrices there is at least one eigenvector $z \in \mathbb{C}^n$, with eigenvalue λ say, for A i.e. $Az = \lambda z$. Rescaling, we may assume that $\|z\|^2 = \langle z, z \rangle = 1$. Let $V = z^\perp = \{x \in \mathbb{C}^n \mid \langle x, z \rangle = 0\}$. Then the fact that $A \in U(n)$ implies that $A(V) \subset V$. Set $u_n = z$ and $\lambda_n = \lambda$. As in the proof of Theorem 5.1, if A' is the restriction of A to V, then $A' \in U(n-1)$ and

$$A = \begin{pmatrix} A' & 0 \\ 0 & \lambda_n \end{pmatrix}.$$

By induction, A' can be diagonalized and all eigenvalues of A' have modulus 1. Moreover, the corresponding eigenvectors $u_1, \ldots, u_{n-1}$ are orthonormal with respect to $\langle -,- \rangle$ Let us check that $|\lambda_n| = 1$. We have

$$1 = \langle z, z \rangle = \langle Az, Az \rangle = \langle \lambda z, \lambda z \rangle = \lambda\overline{\lambda}\langle z, z \rangle = |\lambda|^2.$$

Finally, we should check that there is some $g \in U(n)$ such that $gAg^\dagger$ is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. We just take g to be the matrix with columns $u_1, \ldots, u_n$. If $e_1, \ldots, e_n$ is the standard orthonormal basis of $\mathbb{C}^n$, then

$$\langle ge_i, ge_j \rangle = \langle u_i, u_j \rangle = \delta_{i,j} = \langle e_i, e_j \rangle$$

which implies that $g \in U(n)$. By construction, $gAg^\dagger$ is the required diagonal matrix.

□

In the case $A \in O(n, \mathbb{R}) \subset U(n)$, the fact that the coefficients of the characteristic equation $\det(A - t\mathrm{Id})$ are real implies that if λ is an eigenvalue for A, so too is its complex conjugate. Thus, if $A \in O(3, \mathbb{R})$, there exists $g \in U(3)$ and $\theta \in \mathbb{R}$ such that

$$gAg^\dagger = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & e^{\sqrt{-1}\theta} & 0 \\ 0 & 0 & e^{-\sqrt{-1}\theta} \end{pmatrix}.$$

In particular, if $A \in SO(3, \mathbb{R})$ then A has an eigenvalue equal to one. If it has more than one eigenvalue equal to one, then A will be the identity matrix. Therefore, without loss of generality, the real matrix $\mathrm{Id} - A$ has a one-dimensional kernel $\ell \subset \mathbb{R}^3$. Let $u_1 \in \mathbb{R}^3$ span the kernel. Rescaling if necessary, we may assume that $||u_1|| = 1$. Since A is orthogonal, $(Av, Aw) = (v, w)$ for all $v, w \in \mathbb{R}^3$. In particular, this implies that $A(\ell^\perp) \subset \ell^\perp$, where $\ell^\perp = \{v \in \mathbb{R}^3 \mid (u_1, v) = 0\}$. Fix an orthonormal basis u_2, u_3 of $\ell^\perp$. Then $\{u_1, u_2, u_3\}$ is an orthonormal basis of $\mathbb{R}^3$. With respect to this basis,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix},$$

where $A' \in SO(2, \mathbb{R})$. The only thing left to show is that A' acts as a rotation of the plane $\mathbb{R}^2$ spanned by $\{u_2, u_3\}$. But it is an easy direct calculation to show that $B \in GL(\mathbb{R}^2)$ is a rotation if and only if $B \in SO(2, \mathbb{R})$. This completes the proof of (12).

Finally, we note:

Proposition 5.3. *Let P be a Platonic solid. Then $W(P)$ is a finite subgroup of $O(3, \mathbb{R})$.*

Proof. We have already seen that $W(P)$ is a finite group. Rescaling P if necessary, we may assume that the vertices of P all lie in the 3-sphere $S^3 = \{v \in \mathbb{R}^3 \mid \|v\| = 1\}$. Notice that every transformation $g \in W(P)$ preserves S^3. Therefore it suffices to show that the symmetry group $W(S^3)$ of S^3 equals $O(3, \mathbb{R})$. It's just as easy to consider $W(S^n) \subset GL(\mathbb{R}^n)$. Since $O(n, \mathbb{R})$ can be defined as those $g \in GL(\mathbb{R}^n)$ such that $(g \cdot v, g \cdot w) = (v, w)$ for all $v, w \in \mathbb{R}^n$, we see that $\|g \cdot v\| = \|v\| = 1$ for all $v \in S^3$. Thus, $O(n, \mathbb{R}) \subset W(S^n)$. To show the opposite inclusion, it suffices to show that if u, v are a pair of orthonormal vectors in $\mathbb{R}^n$ i.e. $(v, v) = (u, u) = 1, (u, v) = 0$, then $(g \cdot u, g \cdot u) = (g \cdot v, g \cdot v) = 1$ and $(g \cdot u, g \cdot v) = 0$ for all $g \in W(S^n)$. The first two conditions are automatic. To show that $(g \cdot u, g \cdot v) = 0$, we expand

$$\begin{aligned} 2 &= (u - v, u - v) = (g \cdot (u - v), g \cdot (u - v)) \\ &= (g \cdot u, g \cdot u) + (g \cdot v, g \cdot v) - 2(g \cdot u, g \cdot v) \\ &= 2 - 2(g \cdot u, g \cdot v) \end{aligned}$$

Thus, $(g \cdot u, g \cdot v) = 0$ as required. □

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