LONDON MATHEMATICAL SOCIETY
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NEWSLETTER

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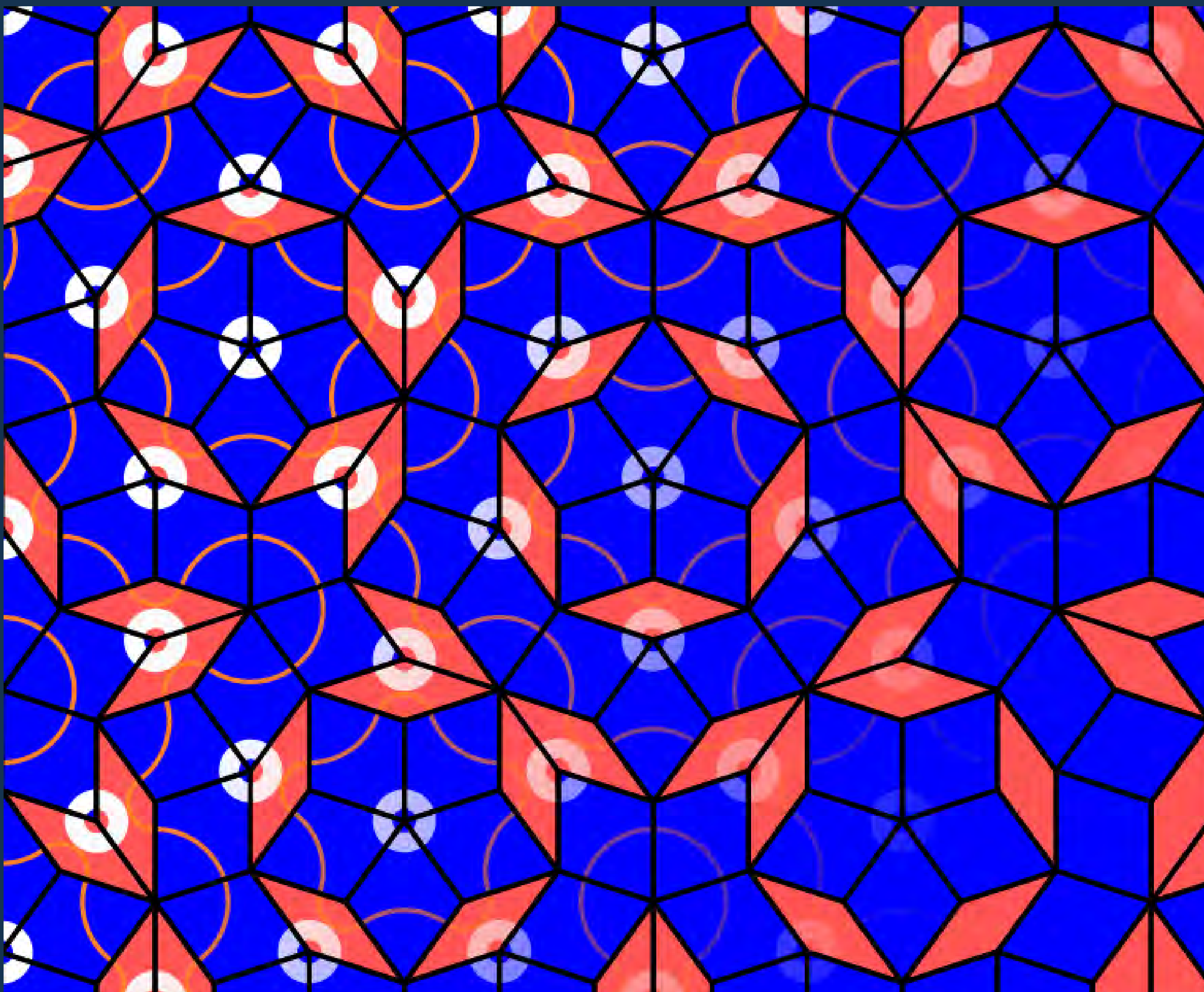

MATHEMATICS OF NEUTRAL INCLUSIONS

INFINITE JIGSAWS
UPSIDE-DOWN FRACTALS
IRRATIONAL SLICES

WINNING AND CODING NIM GAMES

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COVER IMAGE

A finite patch of a Penrose rhomb tiling, from 'Infinite Jigsaws, Upside-Down Fractals and Irrational', by Jamie Walton (page 22).

Do you have an image of mathematical interest that may be included on the front cover of a future issue? Email images@lms.ac.uk for details.

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NEWSLETTER WEBSITE

The Newsletter is freely available electronically at lms.ac.uk/publications/lms-newsletter.

MEMBERSHIP

Joining the LMS is a straightforward process. For membership details see lms.ac.uk/membership.

SUBMISSIONS

The Newsletter welcomes submissions of feature content, including mathematical articles, career related articles, and microtheses from members and non-members. Submission guidelines and LaTeX templates can be found at lms.ac.uk/publications/submit-to-the-lms-newsletter.

Feature content should be submitted to the editor-in-chief at newsletter.editor@lms.ac.uk.

News items should be sent to newsletter@lms.ac.uk.

Notices of events should be prepared using the template at lms.ac.uk/publications/lms-newsletter and sent to calendar@lms.ac.uk.

For advertising rates and guidelines see lms.ac.uk/publications/advertise-in-the-lms-newsletter.

CONTENTS

LMS NEWS

LMS Prizes 2023: Call for Nominations

Deadline: 8 January 2023 (11:59pm)

The LMS invites nominations for the following prizes in 2023, which are intended to recognise and celebrate achievements in and contributions to mathematics:

- The Polya Prize, which is awarded in recognition of outstanding creativity in, imaginative exposition of, or distinguished contribution to, mathematics within the United Kingdom;
- The Senior Whitehead Prize, which is awarded for work in, influence on or service to mathematics, or recognition of lecturing gifts in the field of mathematics;
- The Senior Anne Bennett Prize, which is awarded for work in, influence on or service to mathematics, particularly in relation to advancing the careers of women in mathematics;
- The Naylor Prize and Lectureship in Applied Mathematics, which is awarded for work in, and influence on, and contributions to applied mathematics and/or the applications of mathematics, and lecturing gifts;
- The Berwick Prize, which is awarded to the author(s) of an outstanding piece of mathematical research published by the Society in the past 8 years. The awardee should have fewer than 15 years' full-time involvement in mathematics; and
- The Whitehead Prizes, which are awarded for work in and influence on mathematics to mathematicians with fewer than 15 years' experience at post-doctoral level (up to six may be awarded).

We strongly encourage nominations for all prizes for women and other underrepresented groups in the mathematical community. The Prizes Committee interprets the criteria for all prizes broadly, so if in doubt please submit a nomination.

To submit a nomination for the Polya, Senior Whitehead, Naylor, Senior Anne Bennett, or Whitehead Prize, see: tinyurl.com/prizes2023.

To submit a nomination for the Berwick Prize, see: tinyurl.com/berwick2023.

Full regulations for each prize can be found at lms.ac.uk/prizes/lms-prize-regulations.

Any queries should be sent to James Taylor, Society Governance Officer, at prizes@lms.ac.uk.

The deadline for nominations is 8 January 2023 (11:59pm). Any nominations received after that date will be considered in the next award round.

National Academy of Mathematical Sciences: Setup Phase

The LMS is delighted to inform members of an important step towards the creation of a National Academy for the Mathematical Sciences, with the appointment of the Executive Committee and an Executive Director to carry out the feasibility and setup phase. The proposed academy would represent and advocate for the mathematical sciences and people who work in them including educators, practitioners, and academics, and would operate across the whole of the mathematical sciences, including pure and applied mathematics, statistics and operational research. It would have equality, diversity and inclusivity at its heart.

As part of the Council for Mathematical Sciences, the LMS has been at the heart of the progression of this project and we are delighted that former LMS President, Terry Lyons, FRS, has been appointed as one of the members of the Executive Committee (see cms.ac.uk/wp/executive-committee-biographies). The Executive Committee also includes two current members of Council: LMS Vice-President Cathy Hobbs and Member-at-Large for Women and Diversity Sara Lombardo. The Committee will lead a focussed programme of work, to put in place the structures required for a National Academy to be set up and able to begin its work by the first half of 2025. There will be consultation and discussion in the mathematical sciences community at stages before then. Further details about the proto-Academy can be found at cms.ac.uk/wp/national-academy.

LMS Publications Strategic Retreat

Publications Committee held a strategic retreat at De Morgan House on 13–14 June 2022 to consider medium and long-term plans for the full range of the LMS's publications. Attendees from the Committee and beyond included a wide range of mathematicians including LMS President Ulrike Tillmann and various officers, external experts, and staff from Wiley, one of our publishing partners.

The first day's discussions were driven by the question "Where do we want to be in 10 years' time?" and the second by "How do we get there?". This latter question is likely to be subject to a tricky passage through the rocky and unknown waters of the transition to Open Access and shifts in attitudes around scholarly publishing.

Broadly, the group wanted the Society's current journals to continue to be thriving and highly esteemed, but also to be generating income for the Society's grants and other charitable activities, which can, under the current business models, only be achieved through publishing more articles. Various ideas for doing so were suggested, discussed, and challenged, and will be turned into an action plan by the Committee this autumn.

Niall MacKay
LMS Publications Secretary

Caroline Wallace Leaving as the Society's Executive Secretary

Readers will recall in the May 2022 edition of the Newsletter that Caroline Wallace reflected on the two years she had spent at the Society as its Executive Secretary. Caroline very kindly agreed to stay on longer than the length of her notice period, in order that transitional arrangements could be put in place until the new Executive Secretary took up the position at the beginning of October.

But of course Caroline's leaving could not be put off indefinitely and so on 21 July the Society's staff went out after work for drinks and other light refreshments, to wish her well. We went to Truckles in Bloomsbury, where many recent LMS leaving 'do's' have taken place (pictured). Most LMS staff were present, including John Johnston, the recently departed Communications Officer; Katherine Wright, Society Business, Research & Communications Officer, who was at home that day, fittingly managed to attend the presentation via FaceTime.

Elizabeth Fisher, Membership & Grants Manager, gave a short speech on behalf of the Society's staff, thanking Caroline for her leadership, guidance and support. She highlighted Caroline's involvement in the Protect Pure Maths Campaign, receipt of the Alan Turing £50 note from the Bank of England and the Society's response to the war in Ukraine, which will mean that a generation of mathematicians are safe and still research-active. Although staff over the last two years spent many months apart, Caroline's encouragement to share details of their work regularly, chat together virtually and be reminded of achievements in her weekly staff update meant that we never really felt far apart.

A presentation was made of a metallic Georg Jensen Mirror Bowl, a bottle of Churchill champagne and a photograph of the staff who were present at the Society's drinks reception before the Annual Dinner in November 2021, as well as a card with the best wishes of all the staff.

Caroline in her remarks noted that her time at the Society had been marked (but not marred) by some truly world-changing events, not least the Covid 19 pandemic. Shamelessly stealing a quotation from someone else, Caroline said that we can't choose the times we live in, "all we have to decide is what to do with the time that is given to us". Caroline concluded that she could not have asked for a better group of people with whom to weather the storm.

I'm sure I speak for all the LMS staff when I say that we will miss Caroline and wish her the very best for the future.

James Taylor
Society Governance Officer

Forthcoming LMS Events

The following events will take place in forthcoming months:

LMS Society Meeting and Mary Cartwright Lecture: 10 November, ICMS, Edinburgh and online (bit.ly/3EhUCmK)

LMS/BCS-FACS Evening Seminar: 17 November 2022, online (bit.ly/3C5TX56)

LMS Annual General Meeting & Naylor Lecture 2022: 18 November, Goodenough College, London and online (bit.ly/3C7H2B6)

LMS South West and South Wales Regional Meeting: 17 January 2023, Southampton (bit.ly/3QrQwvy)

LMS Midlands Regional Meeting: 27 March, Warwick (bit.ly/3RW0GVj)

LMS Meeting at BMC 2023: 4 April, Bath

A full listing of upcoming LMS events can be found on page 47.

OTHER NEWS

European Women in Mathematics

European Women in Mathematics (EWM; europeanwomeninmaths.org) is an association of female mathematicians involved in policy and strategic work and in promoting the role of women in mathematics. It was founded in 1986, with one of the cofounders former LMS president Caroline Series, and currently has members in at least 34 countries in Europe. Activities include biennial meetings, annual EWM/EMS summer schools, a mentoring programme, a job board, a programme of travel grants, a network of country coordinators to transfer best practice, and an active mailing list, as well as advocating on behalf of women in mathematics. All LMS members are welcome to join the EWM. You can pay your membership fee with your LMS subscription, by adding EWM membership as a subscription in your LMS account. Departments are also encouraged to join EWM as an institutional member, see tinyurl.com/va7mtu2, and to advertise their vacancies on the job board.

Asian and Oceanian Women in Mathematics

The formation of the Asian and Oceanian Women in Mathematics (AOWM) group was officially established in its First General Assembly, held online on 1 August 2022. The Assembly was attended by more than a hundred participants from Asia and Oceania, among whom are the regional ambassadors of the IMU Committee of Women in Mathematics (CWM) and the founding members of AOWM. The meeting was chaired by Professor Motoko Kotani of Japan.

The establishment of AOWM was announced by Professor Kotani in her welcome message to the Assembly. This was then followed by a congratulatory message given by Professor Marie-Françoise Roy, Chair of the Central Asian Women in Mathematics Association (CWM). Professor Park proceeded to present messages of greetings and congratulations from the presidents of the African Women in Mathematics Association, the Association for Women in Mathematics, European Women in Mathematics, Australian Mathematical Society, Chinese Mathematical Society, Indian Mathematical Society, Indian Women in Mathematics, Indonesian Mathematical Society, Iranian Mathematical Society, Korean Mathematical Society, Korean Women in Mathematical Sciences, Malaysian Mathematical Sciences Society, Mathematical Society of Japan, Mathematical Society of the Philippines, Nepal Mathematical Society, New Zealand Mathematical Society, Ramanujan Mathematical Society, Statistical Society of Australia, the Mathematics Consortium, and Vietnam Mathematical Society.

In this Assembly, the body approved new members of AOWM and the AOWM co-ordinators from countries in the region. The body also chose the following mathematicians as members of the AOWM Executive Committee EC:

President: Sanoli Gun (India)
Vice President: Melissa Tacy (New Zealand)
Vice President: Polly W. Sy (Philippines)
Secretary: Hyang-Sook Lee (Korea)
Other EC members:
Dongmei Xiao (China)
Budi Nurani Ruchjana (Indonesia)
Zohreh Mostaghim (Iran)
Yukari Ito (Japan)
Bakhyt Alipova (Kazakhstan)

A special lecture was given by Professor Nalini Joshi, a member of IMU Executive Committee, titled *Reflections on Gender Diversity in Mathematics*.

Prior to the establishment of AOWM, a Working Group was formed to discuss its guidelines and by-laws and do the preparatory work leading to the First General Assembly. The people behind the Working Group are Motoko Kotani (Chair, Japan), Kyewon Koh Park (Korea), Catherine Greenhill (Australia), Sanoli Gun (India), Polly W. Sy (the Philippines), Dongmei Xiao (China), Bakhyt Alipova (Kazakhstan).

OPPORTUNITIES

LMS Research Schools and Research Schools in Knowledge Exchange 2024

Grants of up to £15,000 are available for LMS Research Schools, one of which will be focused on Knowledge Exchange. The LMS Research Schools provide training for research students in contemporary areas of mathematics. The Knowledge Exchange Research Schools will primarily focus on Knowledge Exchange and can be in any are of mathematics.

The LMS Research Schools take place in the UK and support participation of research students from both the UK and abroad. The lecturers are expected to be international leaders in their field. The LMS Research Schools are often partially funded by the Heilbronn Institute for Mathematical Research (heilbronn.ac.uk) and UK Research and Innovation (ukri.org). Information about the submission of proposals can be found at tinyurl.com/ychr4lwm along with a list of previously supported Research Schools. Applicants are strongly encouraged to discuss their ideas for Research Schools with the Chair of the Early Career Research Committee, Professor Chris Parker (research.schools@lms.ac.uk) before submitting proposals. Proposals should be submitted to Lucy Covington (research.schools@lms.ac.uk) by 22 February 2023.

Clay Mathematics Institute Enhancement and Partnership Program

To extend the international reach of the Research School, prospective organisers may also wish to consider applying to the Clay Mathematics Institute (CMI) for additional funding under the CMI's Enhancement and Partnership Program. Further information about the Program can be found at tinyurl.com/y72byonb. Prospective organisers are advised to discuss applications to The Program as early as possible by contacting the CMI President, Martin Bridson (president@claymath.org). There is no need to wait for a decision from the LMS on your Research School application before contacting the CMI about funding through the Program.

Cecil King Travel Scholarship

The London Mathematical Society invites applicants for the Cecil King Travel Scholarship 2022–23. Applicants should be mathematicians who are registered for a doctoral degree or have completed one in the United Kingdom or the Republic of Ireland within last 12 months.

Two successful applicants will be awarded £6,000 each to support their study or research abroad for a period of three months. The awards are based on written proposals that describe the intended programme of a study or research and the benefits to be gained from such a visit.

One Scholarship is usually awarded to a mathematician in any area of mathematics and one to a mathematician whose research is applied in a discipline other than mathematics.

The deadline to submit an online application is 15 November 2022 and shortlisted applicants will be invited to interview in January 2023. For further details visit bit.ly/3S35RT9.

LMS Undergraduate Research Bursaries in Mathematics 2023

The Undergraduate Research Bursary scheme provides an opportunity for students in their intermediate years to explore the potential of becoming a researcher. The award provides support to a student undertaking a 6–8 week research project over summer 2023, under the direction of a project supervisor.

Students must be registered at a UK institution for the majority of their undergraduate degree and may only take up the award during the summer vacation between the intermediate years of their course. Students in the final year of their degree intending to undertake a taught Master's degree immediately following their undergraduate degree may also apply. Applications must be made by the project supervisor on behalf of the student.

For further information contact Lucy Covington (urb@lms.ac.uk). Application deadline: Tuesday 1 February 2023.

LMS–Bath Symposia 2024: Call for Proposals

Deadline: 15 December 2022

A reminder that proposals are invited from the mathematical community to organise the LMS-Bath Mathematical Symposia, which will be held at the University of Bath in 2024. It is intended that there will be two LMS-Bath Mathematical Symposia and one of the LMS-Bath Symposia will be funded by the Isaac Newton Institute. Core funding at approximately £40,000.00 is available to support each Symposium.

For further details about the LMS-Bath Mathematical Symposia, please visit the Society's website: lms.ac.uk/events/mathematical-symposia or the LMS-Bath Symposia's website: bathsymposium.ac.uk

Before submitting, organisers are welcome to discuss informally their ideas with the Chair of the Research Grants Committee, Professor Andrew Dancer (grants@lms.ac.uk).

Prospective organisers should send a formal proposal to the Grants Team (grants@lms.ac.uk) by 15 December 2022. Proposals are approved by the Society's Research Grants Committee after consideration of referees' reports.

LMS Grant Schemes

The next closing date for research grant applications (Schemes 1, 2, 4, 5, 6 and AMMSI) is 22 January 2023. Applications are invited for the following grants to be considered by the Research Grants Committee at its February 2023 meeting. Applicants for LMS Grants should be mathematicians based in the UK, the Isle of Man or the Channel Islands. For grants to support conferences/workshops, the event must be held in the UK, the Isle of Man or the Channel Islands:

Conferences (Scheme 1)

Grants of up to £5,500 are available to provide partial support for conferences. This includes travel, accommodation and subsistence expenses for principal speakers, UK-based research students, participants from Scheme 5 countries and Caring Costs for attendees who have dependents.

Visits to the UK (Scheme 2)

Grants of up to £1,500 are available to provide partial support for a visitor who will give lectures in at least three separate institutions. Awards are made to the host towards the travel, accommodation and subsistence costs of the visitor. Potential applicants should note that it is expected the host institutions will contribute to the costs of the visitor. In addition, the Society allows a further amount (of up to £200) to cover Caring Costs for those who have dependents.

Online Lecture Series (Scheme 3)

Grants of up to £1,000 are available per year to provide support to mathematicians, or groups of mathematicians, delivering online lecture series in mathematics. Applications for this element of the Scheme 3 grant is open to both Joint Research Groups (new and current) and to mathematicians who are not part of a Joint Research Group.

Research in Pairs (Scheme 4)

For those mathematicians inviting a collaborator, grants of up to £1,200 are available to support a visit for collaborative research either by the grant holder to another institution abroad, or by a named mathematician from abroad to the home base of the grant holder. For those mathematicians collaborating with another UK-based mathematician, grants of up to £600 are available to support a visit for collaborative research either by the grant holder to another institution or by a named mathematician to the home base of the grant holder. In addition, the Society allows a further amount (of up to £200) to cover Caring Costs for those who have dependents.

Research Reboot (Scheme 4)

Grants of up to £1,000 are available to provide support to mathematicians who have found themselves without the time to engage in research due to illness, caring responsibilities, increased teaching or administrative loads or any other factors. The grant offers funding to cover travel, accommodation, subsistence and caring expenses so the applicants can leave their usual environment to focus entirely on research for a period from two days to a week, in order to restart their research activity.

Collaborations with Developing Countries (Scheme 5)

For those mathematicians inviting a collaborator to the UK, grants of up to £3,000 are available to support a visit for collaborative research, by a named mathematician from a country in which mathematics could be considered to be in a disadvantaged position, to the home base of the grant holder. For those mathematicians going to their collaborator's institution, grants of up to £2,000 are available to support a visit for collaborative research by the grant holder to a country in which mathematics could be considered to be in a disadvantaged position. Applicants will be expected to explain in their application why the proposed country fits the circumstances considered eligible for Scheme 5 funding. In addition, the Society allows a further amount (of up to £200) to cover Caring Costs for those who have dependents. Contact the Grants team if you are unsure whether the proposed country is eligible, or check the IMU's Commission for Developing Countries definition of developing countries (tinyurl.com/y9dw364o).

Research Workshop Grants (Scheme 6)

Grants of up to £10,000 are available to provide support for Research Workshops. Research Workshops should be an opportunity for a small group of active researchers to work together for a concentrated period on a specialised topic. Applications for Research Workshop Grants can be made at any time but should normally be submitted at least six months before the proposed workshop.

African Mathematics Millennium Science Initiative (AMMSI)

Grants of up to £2,000 are available to support the attendance of postgraduate students at conferences in Africa organised or supported by AMMSI. Application forms for LMS-AMMSI grants are available at ammsi.or.ke.

The next closing date for early career research grant applications (Schemes 8-9 and ECR Travel Grants) is 22 February 2023. Applications are invited for the following grants to be considered by the Early Career Research Committee at its March 2023 meeting:

Postgraduate Research Conferences (Scheme 8)

Grants of up to £2,500 are available to provide partial support for conferences, which are organised by and are for postgraduate research students. The grant award will be used to cover the costs of participants. In addition, the Society allows the use of the grant to cover Caring Costs for those who have dependents.

Celebrating new appointments (Scheme 9)

Grants of up to £400–£500 are available to provide partial support for meetings to celebrate the new appointment of a lecturer at a university. Potential applicants should note that it is expected that the grant holder will be one of the speakers at the conference. In addition, the Society allows the use of the grant to cover Caring Costs for those who have dependents.

ECR Travel Grants

Grants of up £500 are available to provide partial travel and/or accommodation support for UK-based Early Career Researchers to attend conferences or undertake research visits either in the UK or overseas.

For full details of these grant schemes, and to find information on how to submit application forms, visit the LMS website: lms.ac.uk/content/research-grants. Queries regarding applications can be addressed to the Grants Administrator Lucy Covington (020 7927 0807, grants@lms.ac.uk), who will be pleased to discuss proposals informally with potential applicants and give advice on the submission of an application.

LMS Early Career Fellowships 2022–23

The Early Career Fellowships offer short-term support for early career mathematicians in the transition period between completing their PhD and securing a first postdoctoral position.

How you will benefit

The programme is based on a research focussed visit to an institution during which you are expected to spend your working time on study and mathematical investigation. In addition, you should benefit from professional development and have an opportunity to build connections with other researchers in order to establish and develop a strong academic network. The fellowships may be held at one or more institutions, but not normally at the institution where you studied for your PhD.

How you will be supported

The London Mathematical Society will offer fellowships of between 3 and 6 months to applicants who have recently or will shortly receive their PhD. The award will include a monthly stipend of £1,475 plus travel allowance of £800.

The Early Career Fellowships are partially supported by Heilbronn Institute for Mathematical Research (HIMR) through the UKRI/EPSRC Additional Funding Programme for Mathematical Sciences. In 2021–2022 ten Early Career Fellows were sponsored through this scheme.

How to apply

The deadline for applications for the 2022–23 round is 14 January 2023. For further details about the Early Career Fellowships, including an application form and testimonials from previous fellows, please visit the Society's webpage: lms.ac.uk/grants/lms-early-career-fellowships where a list of current fellows is also available. If you require any further help, please contact us on fellowships@lms.ac.uk.

Valeriya Kolesnykova
Membership, Fellowships & Accounts Assistant

Mathematics PhD and Masters Scholarships

The Martingale Scholarship programme provides full funding to pursue a master's or PhD in mathematics, or an integrated PhD, and a tax-free stipend equivalent to an annual salary of up to £25,000. Embark on your research journey at one of the programme's partner universities: Cambridge, Imperial, King's, Oxford or UCL. Recipients will also receive access to career development opportunities through the Martingale Development Programme.

This programme is open to final year undergraduates and graduates in mathematics or a closely-related degree. More information on the eligibility criteria can be found at martingale.foundation. To apply, you will need to submit a supporting statement, your latest transcript and an academic reference. The deadline for applications is 28 November 2022.

If you would like to know more about Martingale and receive support on your application or ask any questions, email hello@martingale.foundation. Applications can be made at martingale.foundation/apply.

The Heilbronn Institute for Mathematical Research offers two funding schemes for UK mathematics.

Heilbronn Focused Research Grants (FRG)

A number of grants of up to **£8,000** are available to fund focused research groups to work on adventurous and challenging mathematical problems, and/or to discuss important new developments in mathematics.

Groups will normally comprise 8 or fewer people, and will include international participants together with a substantial UK-based membership. Proposals from these areas of research, interpreted broadly, will be given priority: Pure Mathematics, Probability and Statistics, and Quantum Information. These grants will support travel and local expenses, and are open to all mathematicians based at UK universities.

How to apply: Submissions of one A4 page should be sent by **9am** on **Thursday 26th January 2023** to heilbronn-manager@bristol.ac.uk

Heilbronn Small Grants Scheme (SGS)

Grants are available for projects supportive of UK mathematical research. The normal funding limit is **£5,000**, but applications up to **£10,000**, or occasionally greater, may be considered when a strong case is made.

Such grants will support UK-based activities involving mathematical research including, but not limited to, the following purposes:

- Partial or full support towards workshops, small conferences, symposia, colloquia, or seminars.
- Travel and local expenses for visiting professors/academics, PhD students or speakers.
- Support for collaborative research programmes or activities.
- Activities supporting the education and training of students and young mathematicians.
- Support for activity that is not eligible for funding from other sources or where funding is limited.

We anticipate receiving proposals from UK-based mathematicians at all career stages

How to apply: Proposals should be sent to heilbronn-manager@bristol.ac.uk. For applications for £5,000 or less, no more than one page of A4 is needed, and two pages will suffice for most larger applications. There is no deadline, and applications will be considered on a rolling basis.

Finally,

Grants will be funded either through the UKRI/EPSRC 'Additional Funding Programme for Mathematical Sciences' Programme or by the Heilbronn Institute directly. Details of past events may be found at https://heilbronn.ac.uk/events/past/. For further particulars and additional information concerning Heilbronn funding opportunities, please visit our website: http://heilbronn.ac.uk/opportunities/

Annual LMS Membership 2022–23: Reminder

Members are reminded that their annual membership fees for the period November 2022 – October 2023 become due on 1 November 2022 and should be paid no later than 1 December 2022.

Membership subscriptions can be paid online via the Society's website: lms.ac.uk/user. Further information about subscription rates for 2022–23 and a subscription form may be found at bit.ly/3euCHyQ.

Elizabeth Fisher
Membership & Grants Manager

Maximising your Membership: Signing the LMS Members' Book

A unique benefit of LMS membership is the opportunity for members to sign the LMS Members' Book. After election to membership, and once they have paid their subscription fees, members can sign their name into the history of the LMS.

In a leather-bound volume that dates from the Society's founding in 1865, the LMS Members' Book provides a physical link between current members and the first members of the Society, such as Augustus De Morgan, whose signatures appear in the early pages. Leafing through, one finds the signatures of other well-known mathematicians such as M.L. Cartwright, G.H. Hardy, Felix Klein and Henri Poincaré inscribed within the first half of the Members' Book.

While it's been 150 years since the first autographs, just over half the pages have been filled, which leaves plenty of room for any members who have not yet signed to add their name. The LMS Members' Book appears at every in-person Society Meeting, as far as is practicable, and it has travelled across the globe so members based outside the UK are included.

Your next opportunities to sign the LMS Members' Book in 2022-23 will be:

- 18 November 2022; LMS AGM & Naylor Lecture, Goodenough College in London
- 17 January 2023: LMS South West & South Wales Regional Meeting, Southampton University
- 27 March 2023: LMS Midlands Regional Meeting, Warwick University
- 4 April 2023: LMS Meeting at the BMC 2023, Bath University.

At each of these meetings, we will invite those members who can attend in-person, and who have not yet done so, to come forward when asked by the President and sign the LMS Members' Book.

Elizabeth Fisher
Membership & Grants Manager

Report: Aperiodic Tilings: Meeting and Exhibition in Honour of Uwe Grimm

Absorbed in aperiodicity

This meeting and exhibition, which commemorated Professor Uwe Grimm's research and passion for outreach, was a two pronged affair. It consisted of a research meeting, bringing together 50 researchers in aperiodic tiling theory, and an art exhibition and interactive workshop, attracting 750 visitors, of which 450 were school students in Key Stage 3 (years 7-9). The research meeting received funding from an LMS Scheme 1 conference grant, Bielefeld University and the Open University. The art exhibition was funded by an ICMS Public Engagement Activity award. The event website bit.ly/3S43uzG includes an online video of artwork from the exhibit.

Ian Short, Charlotte Webb and Reem Yassawi
The Open University

LEVELLING UP

Here are the latest updates on the Levelling Up: Maths scheme being developed by the LMS, which is made possible by a generous donation from Dr Tony Hill. The scheme seeks to widen participation of those who are under-represented in mathematics. It is part of a broader Levelling Up: STEM project which also covers Physics and Chemistry.

STEM and Black Heritage

Much has been written, over the last twenty years, about the difficulties that young people of Black heritage face in STEM. Yet, despite this awareness, there has been very little progress. Last year, the Royal Society issued a call to action, saying the "STEM sector must step up and end unacceptable disparities in Black staff and students' academic progression and success".

Given the central role of mathematics in STEM, the Institute of Mathematics and its Applications (IMA) and LMS have joined forces and will be offering a new university access programme called *Levelling Up: Maths for Black heritage students*, with support from a broad STEM coalition.

It is envisaged that the first cohort of A level students will apply to join the programme in January 2023 with a programme start in April 2023. The students will be assigned to small virtual tutorial groups and taught and mentored, wherever possible, by Black heritage undergraduates at the participating universities. There are plans to include a variety of influential Black guest speakers including the programme leader, Professor Nira Chamberlain, who commented:

"This is a ground-breaking initiative which builds on the success of Levelling Up: Maths. I would like to thank members of the STEM coalition for their valued support."

This new programme is an unparalleled collaboration between learned societies, educational charities (and non-profits) across STEM and the Black community. In addition to the IMA and LMS, the Royal Academy of Engineering (RAEng), the Chartered Institute for IT (BCS), the Institute of Physics (IOP), the Royal Society of Chemistry (RSC), Mathematics in Education and Industry (MEI) and STEM Learning will promote the programme. Black community support will be provided by the Association for Black and Minority Ethnic Engineers (AFBE-UK) and Black British Professionals in STEM (BBSTEM).

Ulrike Tillmann, LMS President said:

"The LMS is excited about this new development to the very successful Levelling Up: Maths scheme. We know that there is massive untapped mathematical talent among students of Black heritage and are delighted to be able to work with such a strong coalition of partners to bring about change which is so overdue."

Paul Glendinning, IMA President noted:

"The IMA is delighted to be part of the STEM coalition supporting this new initiative. We look forward to seeing it make a positive difference to the futures of all those involved."

The IMA and LMS would like to hear from university Maths departments interested in joining this programme. They should contact either Erica Tyson (Erica.Tyson@ima.org.uk) or Lindsay Walsh (lindsay.walsh@lms.ac.uk) for more information.

Edinburgh Mathematical Society

The Edinburgh Mathematical Society is working with the Levelling Up: STEM team from England to launch a Scottish widening participation programme aimed at underrepresented groups of school pupils in order to encourage them to attend University and to take STEM subjects. The programme, called 'Reaching Higher: Maths' uses undergraduate tutors to provide a mix of mathematical and pastoral support for the school pupils. The pilot of the programme will be launched as part of Maths Week Scotland at the end of September and tutoring sessions will begin in January 2023.

Records of Proceedings at LMS meetings
Women in Astronomy LMS–IMA–BSHM Joint Meeting: 16 September 2022

This meeting was held on Friday 16 September 2022, in person at De Morgan House and virtually, via Zoom, as part of the Joint Meeting with the IMA and BSHM. Over 50 members and guests were present for all or part of the Meeting.

The meeting began at 11.00am with The LMS President, Professor Ulrike Tillmann FRS, Dr Martine J. Barons, CMath, FIMA representing the IMA, and the BSHM President, Sarah Hart, each introducing, in turns, their respective institutions and societies.

There were no members elected to Membership at this Society Meeting.

One member signed the Members' Book and was admitted to the Society.

The President of the Society Lectures and Meetings Committee, Professor Brita Nucinkis, introduced the first lecture given by Dr Meg Schwamb (Queen's University Belfast), on Exploring the Solar System with the Vera C. Rubin Observatory.

After the changeover that followed, Dr Martine Barons introduced the second lecture by Dr Isobel Falconer (University of St. Andrews), on *Maria Cunitz and Urania Propitia – Was Marriage to an Astronomer a 'Career' for Women Mathematicians in the 1600s and 1700s?*.

After the lunch break, Professor Anna L. Watts introduced the third lecture by Professor Suzanne Aigrain (University of Oxford), on *Portraits of Exoplanets on a Red Background*.

After the changeover, Professor Marika Taylor introduced the fourth lecture by Dr Isabelle Lémonon Waxin (Cermes 3), on *She Only Does Calculations Like a Machine ... The French Enlightenment Women Astronomers: Simple Arithmeticians?*.

The fifth lecture, scheduled to be given by Professor Mathilde Jauzac (Durham) on *Uncovering the Nature of the Dark Sides of our Universe using Cosmic Beasts* did not take place, as Professor Jauzac was sick and therefore unable to give the lecture.

Brita Nucinkis, Martine Barons and Sarah Hart then thanked the speakers for their excellent lectures and expressed the thanks of the Society to the Organising Committee for a wonderful meeting.

Afterwards, a wine reception was held at De Morgan House, followed by a Society dinner at the Antalya Restaurant.

The Princess and the Pea: The Mathematics of Neutral Inclusions

ELEANOR RUSSELL AND WILLIAM J. PARNELL

There are boundless real-world examples when a particle, inclusion, or filler is embedded inside another medium. The presence of the inclusion generally leads to the localisation of some field, with a maximum on its interface with the host. This can be problematic, e.g. in composite media where inclusions are introduced to improve a property such as stiffness or conductivity, stress concentrations around the inclusions can lead to material failure in the longer term. *Neutral inclusions* (NIs) are coated particles, with coatings designed to ensure that such concentrations in the host medium disappear and thus render the material more sustainable. Here we discuss the history and motivation of NIs, including their relationship to cloaks. We describe potential applications and, most importantly, how mathematics helps us in the quest to design and manufacture them.

Hans Christian Andersen's fairy tale of the Princess and the Pea describes how the princess proves her royal identity when she is unable to sleep due to a very hard pea located under her many mattresses. As illustrated in Figure 1, despite the distance between the pea and the top mattress, the perturbations caused by the pea extend upwards, meaning that the princess is uncomfortable and cannot sleep.

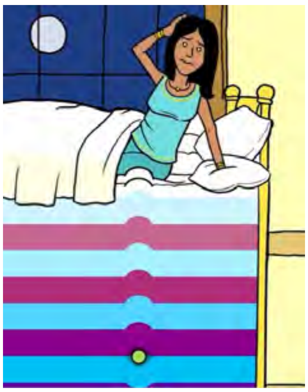

Figure 1. Illustration of the perturbations caused by the pea [1].

The presence of the pea here is analogous to that of an inclusion embedded in a background or matrix medium. Inclusions are ubiquitous in structures, composite materials and metamaterials and are typically introduced to enhance material properties such as stiffness, toughness or thermal or electrical conductivity. However, like the pea, the presence of inclusions disturbs the local stress and strain fields in the matrix medium in which they reside. As a result, stress concentrations can often lead to crack initiation and subsequent global structural failure.

Stress singularities

Stress concentrations give rise to singularities that can have catastrophic consequences - including ships sinking and planes falling out of the sky! A classic example is the world's first jetliner, the Comet, which was designed with large, square windows, see Figure 2(a). The windows and cabin wall here correspond to the notion of inclusions and the matrix respectively. Although not the only reason, the square window shape was a contributing factor in the structural failure of the plane, given the increased stress concentration near the corners. See www.rafmuseum.org.uk.

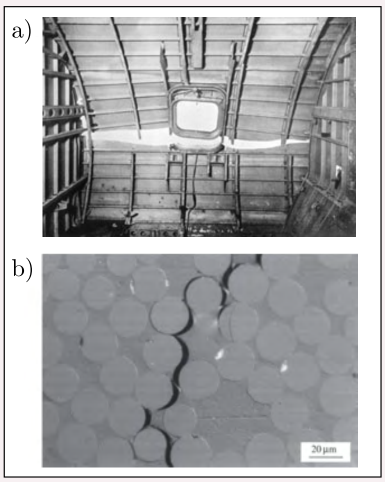

Figure 2. Structural failure in (a) the Comet fuselage (image credit: www.rafmuseum.org.uk) and (b) a transverse crack formed in a fibre-matrix under loading [2].

Numerous mechanisms have been proposed to mitigate problems arising due to stress concentration, such as carefully designing the shape of an inclusion. Indeed, entire books have been dedicated to tabulating stress concentrations around specific structural artefacts [3]. One approach is to introduce a reinforcing coating on the filler surface, the aim of which is to improve the bonding between the matrix and the inclusion. Typically however this results in only modest improvements and stress concentrations remain relatively high [4].

An exciting concept, which has not yet been fully exploited in advanced materials, is that of the neutral inclusion (NI). These coated inclusions, as depicted in Figure 3 for the spherical case, are designed to reinforce an inclusion without altering the stress fields in the matrix when a specific loading type is applied, as if the NI was in fact absent. A neutral pea in Figure 1 for example would not perturb the mattresses around it, thus yielding a more comfortable bed for the princess. Therefore, in principle, NIs enable the redistribution of stress whilst still enhancing the materials design space, including the possibility of lighter, stronger materials.

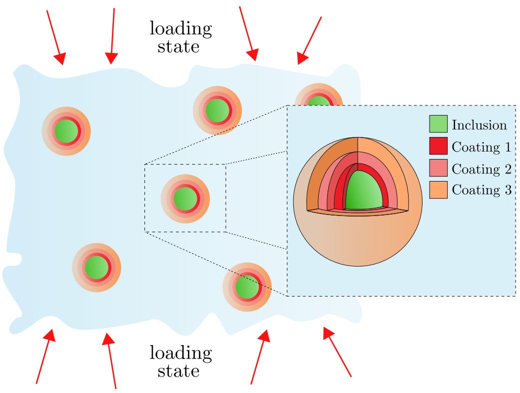

Figure 3. Schematic of a spherical neutral inclusion designed using three coatings.

NIs were first introduced by Mansfield in 1953 in the form of reinforced holes in a plane sheet [5]. Since then they have been considered in many different physical scenarios, although not completely in the context of elastostatics.

To fix on a specific example we shall consider a fibre-reinforced medium consisting of long unidirectional fibres embedded in a matrix, as depicted in Fig. 4(a). The fibres are usually introduced to improve a property such as stiffness or conductivity and they will usually induce anisotropy (directional dependence of the properties) i.e. an increased stiffness in the direction of the axis as compared with the stiffness in the plane of the circular cross-sections of the fibres, depicted in Fig. 4(b). The geometry of such media means that problems are essentially two dimensional. In terms of the NI problem, we shall consider what happens in the vicinity of a coated fibre as illustrated in Fig. 5.

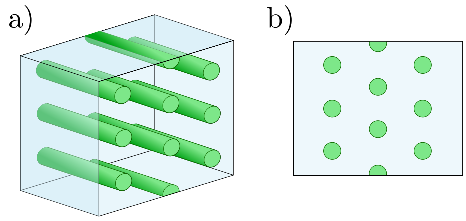

Figure 4. Schematic of (a) a fibre-reinforced medium in 3D and (b) its 2D cross-section.

The two-dimensional equations of elastostatics can be written in the form

$$\mathrm{div}\boldsymbol{\sigma} = 0 \tag{1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor. Given that we are working with fibres of circular cross-section it is convenient to employ circular cylindrical polar coordinates r, θ, use the decomposition

$$\boldsymbol{\sigma}(r,\theta) = \sum_{-\infty}^{\infty} \boldsymbol{\sigma}_n(r)e^{in\theta} \tag{2}$$

and then take the real part of this for practical purposes. We consider a specific n as being associated with a given deformation, for example $n = 0$ corresponds to hydrostatic deformation in the plane. Let us therefore begin with this case, which leads us to a scalar boundary value problem.

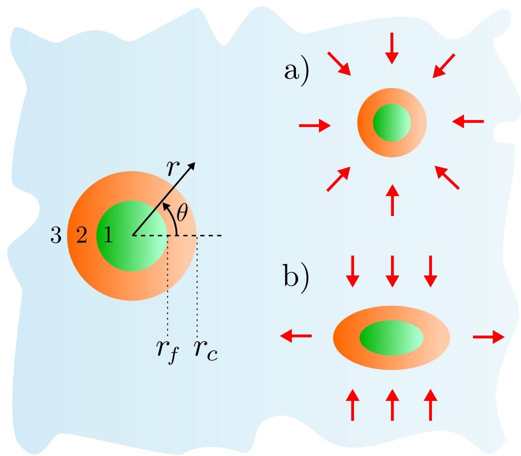

Figure 5. A coated inclusion embedded in a matrix medium subjected to two far-field loading states: (a) hydrostatic compression and (b) pure shear.

Scalar elastostatics - planar hydrostatic loading

The case of $n = 0$ gives rise only to r dependence. This is consistent with planar hydrostatic loading $\sigma_{rr} \to -p$ as $r \to \infty$, for some imposed pressure p, in the xy-plane with the fibre axis parallel to the z-axis, as illustrated in Figure 5(a). In general a problem in elastostatics will be vectorial in nature, but the geometry and loading in this case lead to a scalar problem for the radial displacement $u_r = u_r(r)$. The stress-strain relation is

$$\sigma_{rr}(r) = (K(r) - \mu(r))\frac{du_r}{dr} + (K(r) + \mu(r))\frac{u_r}{r} \tag{3}$$

and the governing equation (1) becomes

$$\nabla \cdot (K(r)\nabla u_r(r)) = 0 \tag{4}$$

where $K = \lambda + \mu$ is the planar bulk modulus (with λ and μ the standard Lamé moduli of linear elasticity), which takes the values

$$K(r) = \begin{cases} K_f, & r < r_f, \\ K_c, & r_f < r < r_c, \\ K_m, & r > r_c. \end{cases} \tag{5}$$

Solutions of (4) take the form

$$u_r(r) = \begin{cases} A_f r, & r < r_f, \\ A_c r + \dfrac{B_c}{r}, & r_f < r < r_c, \\ -\dfrac{p}{K_m} r + \dfrac{B_m}{r}, & r > r_c. \end{cases} \tag{6}$$

Imposing continuity of traction σ_{rr} and displacement u_r at $r = r_f$ and $r = r_c$ yields expressions for A_m, B_m, A_c, B_c and A_f. To ensure neutrality we set $B_m = 0$, meaning that for $r > r_c$, $u_r = -\dfrac{p}{2K_m} r$ and therefore $\sigma_{rr} = -p$ for all $r > r_c$, as would be the case with no inclusion present for this loading type. This requires

$$\left(\frac{r_f}{r_c}\right)^2 = \frac{(K_m - K_c)(K_f + \mu_c)}{(K_f - K_c)(K_m + \mu_c)}. \tag{7}$$

Typically matrix and fibre properties will be specified and then the coating property together with the ratio r_f/r_c can be chosen.

It is worthwhile to note that this problem and the result (7) is analogous to problems in thermal and electrical conductivity, as described in the band box below.

The above illustrates that the neutral coating properties are dependent on those of the matrix and fibre, but also the loading condition. The question thus arises, are are we able to maintain neutrality if we modify the loading state? In particular in the case of elastostatics the second important loading case is that of pure shear, corresponding to $n = 2$ in the modal decomposition (2), see Figure 5(b). This corresponds to a loading state consistent with

$$\sigma_{xx} \to T, \qquad \sigma_{yy} \to -T \tag{8}$$

as $|\mathbf{x}| = r \to \infty$, or equivalently

$$\sigma_{rr} \to T\cos 2\theta, \qquad \sigma_{r\theta} \to -T\sin 2\theta. \tag{9}$$

Can we ensure neutrality for this loading state with properties consistent with (7)? In the next section we summarise the work of [6] in which this question is answered in the affirmative, but only if coating properties are anisotropic. This leads us to the notion of strong and weak neutrality.

Vectorial elastostatics - exploiting anisotropy

In order to solve the $n = 2$ pure shear vectorial problem for radial and azimuthal displacement fields u_r and u_θ we summarise the work in [6], employing the sextic formalism of anisotropic elasticity [7]. We allow the coating to be cylindrically anisotropic, define the impedance matrix $\mathbf{Z}(r)$ via

$$\begin{pmatrix} r\sigma_{rr} \\ -ir\sigma_{r\theta} \end{pmatrix} = \mathbf{Z}(r)\begin{pmatrix} u_r \\ -iu_\theta \end{pmatrix} \tag{10}$$

and solve the problem in terms of the vector $\mathbf{v} = (u_r, -iu_\theta, r\sigma_{rr}, -ir\sigma_{r\theta})$. The impedance matrix depends on the anisotropic elastic properties of the coating and also those of the fibre.

The r-dependent solution of the problem in the isotropic matrix, which satisfies the far field imposed pure shear loading, can be written as [8]

$$\mathbf{v} = b_m r\boldsymbol{\alpha}_2 + \frac{c_m}{r}\boldsymbol{\alpha}_3 + \frac{d_m}{r^3}\boldsymbol{\alpha}_4 \tag{11}$$

where $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_3$ and $\boldsymbol{\alpha}_4$ are known eigenvectors and b_m, c_m and d_m are constants, to be determined. Imposing continuity conditions on the boundary between the coating and matrix on $r = r_c$ yields

$$b_m\boldsymbol{\alpha}_2 + c_m\boldsymbol{\alpha}_3 + d_m\boldsymbol{\alpha}_4 = \begin{pmatrix} \mathbf{U} \\ \mathbf{Z}(r_c)\mathbf{U} \end{pmatrix} \tag{12}$$

where $\mathbf{U} = (u_r(r_c), -iu_\theta(r_c))$.

Neutrality requires $c_m = d_m = 0$ and using the fact that $\boldsymbol{\alpha}_2 = [1, 1, 2\mu_m, 2\mu_m]^T$, we find

$$b_m \begin{pmatrix} 1 \\ 1 \\ 2\mu_m \\ 2\mu_m \end{pmatrix} = \begin{pmatrix} \mathbf{U} \\ \mathbf{Z}(r_c)\mathbf{U} \end{pmatrix} \implies \mathbf{U} = b_m \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{13}$$

So $\mathbf{U}$ is an eigenvector of $\mathbf{Z}(r_c)$ with eigenvalue $2\mu_m$. The condition in (13), implies that $2\mu_m = Z_{11} + Z_{12} = Z_{22} + Z_{21}$. This together with the symmetry $\mathbf{Z} = \mathbf{Z}^T$ leads to the requirement that $Z_{11} = Z_{22}$ and therefore $2\mu_m = Z_{11} + Z_{12}$. This is therefore the neutrality condition relating the properties of the required NI coating to those of the matrix and fibre.

If we assume that the coating is isotropic, it transpires that the neutrality condition above implies $\mu_f = \mu_c = \mu_m$, i.e. we reduce to the trivial homogeneous case. We conclude therefore that we are not able to achieve neutrality with isotropic coatings. If coatings are cylindrically anisotropic however, the condition $2\mu_m = Z_{11} + Z_{12}$ coupled with the hydrostatic condition (7) provides requirements on the coating such that we obtain neutrality. Anisotropy is therefore key to achieving neutrality in elastostatics.

We term the above *strong neutrality*, distinguishing it from a weaker form of neutrality that can be achieved by setting only $c_m = 0$ but not insisting that d_m satisfies any criterion. In this case

$$\begin{pmatrix} \boldsymbol{\alpha}_2 & \boldsymbol{\alpha}_4 & \begin{matrix} \mathbf{I} \\ \mathbf{Z}(r_c) \end{matrix} \end{pmatrix} \begin{pmatrix} b_m \\ d_m \\ -\mathbf{U} \end{pmatrix} = 0. \tag{14}$$

To find solutions we set the determinant of the matrix to zero to find

$$6\mu_m = Z_s + 2Z_{12} + \sqrt{(2Z_s + Z_{12})^2 - 3Z_d^2} \tag{15}$$

where

$$Z_s = \frac{1}{2}(Z_{11} + Z_{22}), \qquad Z_d = \frac{1}{2}(Z_{11} - Z_{22}) \tag{16}$$

thus relating matrix elastic properties to anisotropic coating and fibre properties via the impedance matrix. It transpires that isotropic coatings can ensure both (15) and (7) hold. Thus isotropic coatings can ensure weak neutrality but not strong neutrality.

In Figures 6 and 7 we illustrate neutrality for a specific case of a void-like fibre. The complete details of this case can be found in Example (ii) from Table 1 in [6]. The figures illustrate that the vertical component of displacement u_y and hoop stress $\sigma_{\theta\theta}$ (respectively) remain unchanged in the matrix in the case of pure shear, in the presence of such an appropriately coated (neutral) void, with $r_c = 1$ here. What is also noticeable is that the hoop stress is reduced everywhere in the case of strong neutrality, even inside the coating itself.

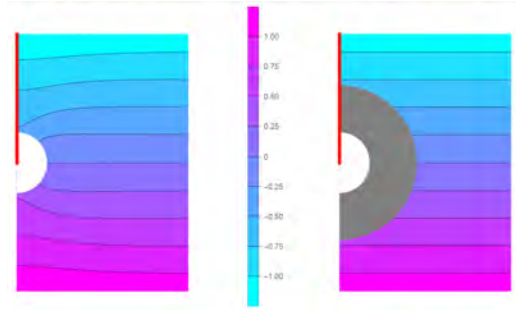

Figure 6. Contours of vertical displacement around a void (left) and neutral void (right) under pure shear loading.

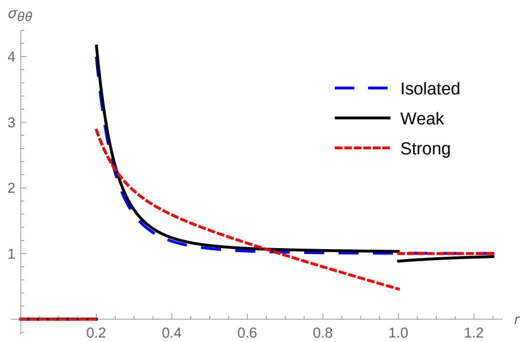

Figure 7. Dependence of hoop stress $\sigma_{\theta\theta}$ on $r > 0$ along the red line $\theta = \pi/2$ in Fig. 6 for the alternative cases of no coating (isolated void), weak neutrality and strong neutrality.

For neutrality subject to even more complex loading conditions ($n \geq 3$), one can envisage that additional anisotropic coatings are required, as illustrated in Figure 3 for a spherical NI. However for practical utility the $n = 0$ and $n = 2$ modes are most important. From a physical perspective, it will not necessarily be the case that for all types of inclusions and matrix media one can find a physically realisable coating that ensures neutrality. But the results above now ensure that engineers and material scientists have conditions and criteria that can be employed to attempt to design and fabricate NIs.

Energetic neutrality and low-frequency transparency

In a well-known paper by Christensen and Lo [8] published in 1979, the *generalised self-consistent method* of micromechanics was developed to yield the effective shear modulus μ_* of composite materials. To do this an identical canonical problem to the NI problem depicted in Figure 5 was considered. Media 1, 2 and 3 correspond to the inclusions, matrix and effective medium respectively. The idea of the method is to choose the effective properties of the exterior/effective medium region in such a way as to render the inclusion/matrix region energetically neutral to the imposed loading. This method uncovered a very useful expression for an estimate of the effective shear modulus of inhomogeneous media, which is still useful today. Its relation to the NI is that energetic neutrality is equivalent to weak neutrality. The approach does not require any condition on d_m and it is therefore non-zero in general, corresponding therefore to weak neutrality.

A further related area is that of low-frequency transparency, in which the leading order field (leading order in the sense that the wavelength is much larger than the scatterer) scattered from an incoming wave such as sound or light, is set to zero via an appropriate choice of coating [9]. One can show that low-frequency transparency is equivalent to energetic (weak) neutrality. Low-frequency transparency is a form of weak cloaking, which leads us on to how neutrality relates to cloaking.

Cloaking

The analogy of neutral inclusions with a metamaterial cloak is a strong one. However, the dependence of NI coating properties on loading type is in contrast to cloak design, where the idea is to employ properties such that concentrations in the host medium disappear for all loading types. The latter is clearly a more difficult task and indeed, transformation-based techniques can be employed, coupled with homogenisation theory of complex microstructures to optimise cloak designs, although these properties are often quite extreme [10]. For that reason a NI is perhaps more practical.

We close by pointing the reader to a video that we made in collaboration with illustrator John Cooper in order to describe the concept of neutral inclusions to a broad audience [1].

FURTHER READING

[1] The Princess and the Neutral Inclusion. [online video] Available at: www.youtube.com/watch?v=04arXGqPHMM
[2] E. K. Gamstedt and B. A. Sjögren, 1999. Micromechanisms in tension-compression fatigue of composite laminates containing transverse plies, Comp. Sci. Tech. 59, 167-178.
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Thermal and electrical neutral inclusions

Neutral inclusions can be employed in many applications outside the realm of elastostatics. In particular, many advances have been made in thermal and electrical applications due to the temperature and electric potential being scalar fields, which greatly simplifies the problem. Perhaps the simplest example is when a temperature gradient is imposed across the matrix by fixing different temperatures along two opposing sides of a sample. Indeed, when these conditions are imposed for a matrix with no inclusions, we recover a simple, linearly varying temperature field, as illustrated in Figure 8(a). Figures 8(b)-(c) go on to show that, although the field is perturbed when inclusions (peas) are embedded in the matrix, we can recover the original linearly varying form by reinforcing each pea with a coating. Thus the combination of the pea and its coating form a NI.

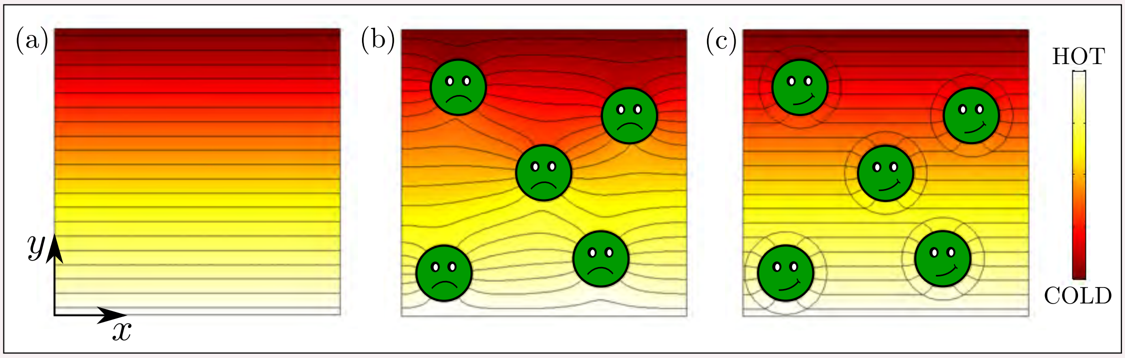

Figure 8. Resulting temperature fields in a matrix with (a) no inclusions, (b) inclusions and (c) neutral inclusions.

To determine the necessary relationship between the thermal conductivities of the matrix, coating and fibre, denoted k_m, k_c and k_f respectively, we solve the problem where temperature fields are fixed to have the desired form and perfect contact conditions (continuity of temperature and heat flux) are imposed at each interface. We consider the case of the fibre-reinforced medium as illustrated in Fig. 4 so that the problem is two-dimensional, with fields being invariant along the axis of the fibres. Then the desired temperature field in the matrix, coating and pea are given by the following solutions of Laplace's equation in two dimensions:

$$\Phi_m(r,\theta) = A_m r \sin\theta, \quad \Phi_c(r,\theta) = \left(A_c r + \frac{B_c}{r}\right)\sin\theta \quad \text{and} \quad \Phi_f(r,\theta) = A_f r \sin\theta \tag{17}$$

respectively, where A_m, A_c, B_c and A_f are constants. From continuity of temperature we obtain $A_f = A_c + B_c/r_f^2$ and $A_c + B_c/r_c^2 = A_m$, where r_f and r_c are the radius of the fibre and coating respectively. In addition, from continuity of heat flux we obtain $k_f A_f = k_m(A_c - B_c/r_f^2)$ and $k_c(A_c - B_c/r_c^2) = k_m A_m$. After some manipulation we find that provided

$$\left(\frac{r_f}{r_c}\right)^2 = \frac{(k_m - k_c)(k_f + k_c)}{(k_f - k_c)(k_m + k_c)} \tag{18}$$

is satisfied, the desired effects of a NI are achieved. To visualise how our choice of conductivities affects the form of the temperature field check out our thermal NI game: scratch.mit.edu/projects/691865184.

Acknowledgements

WJP thanks EPSRC for funding via EP/S019804/01 and EP/V049488/1. ER gratefully acknowledges EPSRC and Thales UK for a PhD CASE award. The authors thank Raphael Assiër for comments on drafts.

William J. Parnell

Will is Professor of Applied Mathematics in the Mathematics of Waves and Materials group at the University of Manchester. His main research interests lie in the development of new mathematical tools and techniques in order to model the behaviour of complex materials and metamaterials. He also uses these techniques to design *new* materials with novel properties and engages with experimentalists in order to bring these ideas into fruition where possible. This includes how microstructures of such materials can be fabricated in order to manipulate and control wave propagation. Prior to him having too many commitments to breathe, Will used to play the drums, and was the drummer in the Bristol Creams Big Band when a mathematics undergraduate many moons ago. He has not played for years now but still has his sights set on an alternative career in rock and roll. His wife and kids are less keen.

Eleanor Russell

Eleanor is a PhD student in the Mathematics of Waves and Materials group at the University of Manchester. Her research focuses on employing thermal metamaterials to design heat spreaders. Eleanor takes fancy dress competitions very seriously and can't wait to celebrate completing her PhD with a science-themed party.

Infinite Jigsaws, Upside-Down Fractals and Irrational Slices: Finding Order Without Periodicity

JAMES J. WALTON

Periodic decorations have been familiar to both mathematicians and artists for millennia. However, periodicity is not necessary for long-range order, a fact whose associated mathematical theory was only initiated more recently. This article explores what aperiodic order is, how to construct examples of aperiodic order and how mathematicians can begin studying it.

Honeycombs and crystals

Suppose you have a box of identical squares, that can be clipped together edge-to-edge. It is obvious what happens when we try to cover a large area with them: after placing the first square, others are forced, building a familiar tessellation. The idealised, infinite tiling is periodic, meaning it is invariant under some non-trivial translations. In this way, long-range order is forced from local geometric constraints. This is utilised by interior decorators re-tiling bathrooms, the bees making their beautiful hexagonal honeycombs and artists, such as in M.C. Escher's periodic tessellations or many of the dazzling tilings of the Alhambra.

Periodicity is also apparent in crystals, such as diamonds. The most striking examples reveal aspects of their internal structure at the macroscopic scale, through the orientation of their facets. Although remarkably intricate, it is not too hard to imagine how this could happen, analogously to how we built our square tessellations or the bees fashion their honeycombs: basic cells might only be able to bond in a way that forces a periodic repetition.

A great advance in our understanding of the structure of crystals followed Max von Laue's work on X-ray diffraction. Atoms of the subject scatter X-rays that are measured at a distant screen. They mostly deconstructively interfere but at a few isolated points on the screen, due to the commensurate distances of the scatterers, the waves can constructively sum to so-called Bragg peaks. The pattern of Bragg peaks can reveal the internal symmetries of the structure. Only n-fold symmetry for $n \leq 6$ and $n \neq 5$ can be observed, predicted from the 230 fully classified space groups, those groups of isometries of $\mathbb{R}^3$ preserving a periodic pattern.

These limitations on the possible symmetries of crystals were well understood, but a shockwave was incoming. In 1982 Dan Shechtman found 10-fold rotational symmetry in the electron diffraction pattern of an aluminium–manganese alloy; a qualitatively similar pattern can be seen in Figure 2.

Forbidden symmetries

Regular planar tilings of squares or triangles (alternatively hexagons) show that there are periodic tilings with n-fold symmetry for any $n \leq 6$, $n \neq 5$. To rule out $n = 5$, suppose the blue and red points in Figure 1 are distinct centres of 5-fold rotational symmetry in a periodic planar tiling, with minimal distance between them (if one such point exists then infinitely many do, by periodicity). Rotating the red point $72°$ ($2\pi/5$ radians) anti-clockwise around the blue point, and the blue point $72°$ clockwise around the red point, we find a closer pair with 5-fold symmetry, contradicting our original assumption.

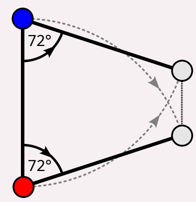

Figure 1. Ruling out 5-fold symmetry

The same argument also prohibits $n > 6$. In dimension 3 there are no new orders of symmetry available, but in dimension 4 there are periodic patterns that have isometries of order 5, 8, 10 or 12.

Note the conspicuous gap of 5-fold symmetry above: periodically repeating patterns in $\mathbb{R}^3$ cannot have point groups with order 5 elements. But if Bragg peaks imply order, order implies periodicity and periodicity implies no 5-fold symmetry, how can the diffraction have 5- or 10-fold symmetry?

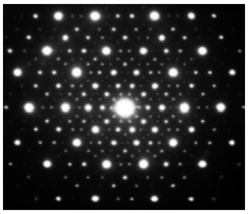

Figure 2. Electron diffraction pattern for icosahedrite

Something had to give. Following heated debate, consensus was eventually reached: order need not imply periodicity. Shechtman's diffraction pattern clearly implied intricate structural order, but periodicity was ruled out by forbidden symmetries.

Materials based on ordered but non-periodic arrangements—in practice given away by forbidden symmetries—are now known as quasicrystals. Shechtman was awarded the Nobel Prize in Chemistry in 2011 for his role in the discovery of them. There were also other experimentalists and theoreticians finding similar structures around the same time, such as the diffraction patterns of sodium carbonate crystals found by de Wolf and van Aalst in 1972 and Mackay's experimental determination of the diffraction of the Penrose tilings in 1977 [7]. These beautiful tilings (see Figure 3) were introduced by Penrose in 1974, discovered by considering how to systematically fill gaps between pentagons. They exhibit 5-fold symmetry and diffraction similar to that in Figure 2, and are central examples in the field of Aperiodic Order, the mathematical study of patterns which are highly ordered without being periodic. Like reliable friends, the Penrose tilings will make frequent showings throughout the article.

Tilings and point sets

A *tile* in $\mathbb{R}^d$ is a compact subset, equal to the closure of its interior (to ensure some regularity; tiles are usually polygons, or at least topological discs). Sometimes it is useful to add colours to tiles, to allow different tile types of the same shape. A *tiling* is then a covering of $\mathbb{R}^d$ by tiles, where distinct tiles intersect on at most their boundaries. We call a finite collection of tiles of a tiling a *patch*, and in particular an r-patch is a patch of tiles intersecting an r-ball.

An alternative way of decorating $\mathbb{R}^d$ is with point patterns, and more specifically Delone sets (point sets where points do not get arbitrarily close, nor leave arbitrarily large holes without points). If desired, points of a Delone set Λ may additionally carry colours. An r-patch is the intersection of Λ with an r-ball. Often (as we shall do henceforth), one considers patches up to translation equivalence, although it is also of interest to allow for general isometries. Many properties considered in Aperiodic Order are 'long-range', and thus invariant under switching between mutually locally derivable (MLD) patterns, which loosely speaking means there are locally defined rules for redecorating one pattern to the other and vice versa. This allows us to work with tilings and Delone sets somewhat interchangeably.

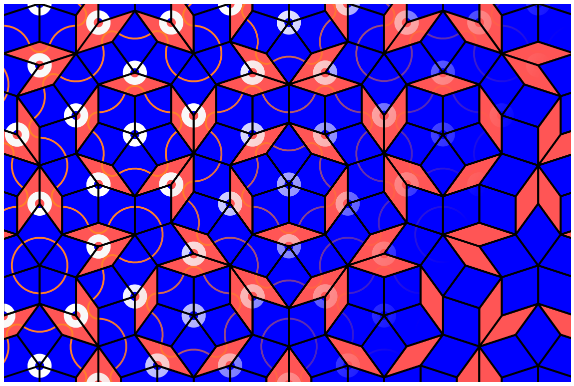

Figure 3. A finite patch of a Penrose rhomb tiling, with (locally derivable) tile decorations faded out on the right

A combinatorial type of condition called finite local complexity (FLC) is satisfied for many important examples, meaning that there are only finitely many r-patches for any given $r > 0$, up to translation equivalence. For tilings, this is equivalent to there being only finitely many tile types, up to translation, and only finitely many ways for two tiles to meet along their boundaries. It is interesting to consider the growth rate of the complexity function $p(r)$, defined as the number of r-patches. Slow growth of $p(r)$ may be considered as a signifier of order; for example, $p(r)$ is bounded if and only if the pattern is periodic.

Diffraction, complexity and repetitivity

A tiling (or Delone set) P is periodic if there is some $x \neq 0$ for which $P + x = P$. Simple enough, but what do we mean by saying that P is *ordered*? There is no single answer to this, partly because different notions are of interest depending on the setting. In quasi-crystallography, for instance, one would like to know when patterns have pure point diffraction.

The diffraction measure

A Delone set Λ defines a Dirac comb $\delta_\Lambda := \sum_{x\in\Lambda}\delta_x$, where δ_x is the Dirac point mass at x. The diffraction measure is the Fourier transform $\widehat{\gamma}$ of the autocorrelation measure γ, defined as an Eberlein convolution

$$\gamma = \lim_{r\to\infty} \frac{\delta_{\Lambda\cap B(r)} * \delta_{-(\Lambda\cap B(r))}}{|B(r)|},$$

provided this limit exists (it does when Λ is sufficiently well ordered). Here, $B(r)$ is the closed ball of radius r, with volume $|B(r)|$. The idea is that, for a finite point set Λ, the Fourier transform $\widehat{\delta_\Lambda} = \sum_{x\in\Lambda}\exp(-2\pi i x\xi)$ models the waves scattered by Λ and the intensity squared of this is equal to the Fourier transform of the unaveraged version of the convolution above; when passing to infinite point sets, we need to normalise the intensities by averaging over increasingly larger areas. One may also allow for varying complex (rather than unit) weights on Λ, in which case some of the definitions above need slight modifications.

The diffraction measure $\widehat{\gamma}$ detects long range correlation of spacing between points of a point set Λ. It uniquely decomposes into pure point, singular continuous and absolutely continuous parts. It is of great interest to identify the spectral types of these diffraction measures for different aperiodically ordered patterns, and in particular find those with only pure point part, so that $\widehat{\gamma}$ is a countable sum of Dirac deltas. This is a mathematical idealisation of a crystal, or quasicrystal, diffracting to sharp Bragg peaks, as opposed to producing a diffuse image.

As well as pure point diffractivity, there are plenty of other senses in which a pattern can be 'ordered'. We've already met one: having low complexity. Another is the pattern's repetitivity. For an FLC pattern, we define the repetitivity function $\rho(r)$ to be the smallest value of R such that every R-ball contains every r-patch. If $\rho(r) < \infty$ for all r then we call the pattern repetitive, that is, if any given finite patch that appears *somewhere* in the pattern in fact appears within a bounded distance from *anywhere* in the pattern.

Slow growth of $\rho(r)$, that is, frequent recurrence of finite patches relative to their size, is another notion of a pattern being ordered. Asymptotically, a non-periodic pattern can be at best *linearly repetitive*, that is, $\rho(r) \ll r$ (meaning $\rho(r) \leq Cr$ for all $r \geq 1$ and some fixed C). Such patterns can be seen as on the periphery of aperiodicity: any more repetitive and they would be forced to be periodic. It may be shown that linearly repetitive patterns of $\mathbb{R}^d$ have complexity $p(r) \asymp r^d$ (that is, $r^d \ll p(r) \ll r^d$), finite patches are forced to occur with well-defined positive frequencies across the pattern and the diffraction measure is well-defined [6].

Infinite jigsaws

How do we construct aperiodic patterns? Finding large classes of repetitive non-periodic patterns is not that easy. Currently, the two main constructions are substitution rules and the cut and project method, which will be discussed later. But before doing so, we turn back to one of the origins of the story.

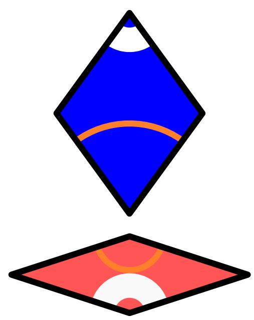

Figure 4. The two Penrose rhombs. Tiles must be placed so that the thin and thick line decorations are carried across edges, as seen on the left of Figure 3

Recall our simple example of squares forcing infinite periodic tessellations, or how the bees are forced to build periodic honeycombs if their cells are all the same hexagonal shape. Congruent copies of a 1–1–$\sqrt{2}$ triangle can be used to build both periodic and non-periodic tilings (by choosing to cut one diagonal or the other of each tile in a periodic square tiling). If one may tile the plane with copies from a finite set of tiles, must it be possible to find a periodic example? This question was posed by Hao Wang in the 1960s for so-called Wang tiles, which are unit squares with coloured edges. A finite set of Wang tiles is said to admit tilings if it is possible to cover the plane with translated copies of them, matching edge-to-edge with agreeing colours. If any finite set of Wang tiles which admits tilings also admits periodic tilings, that would lead to a simple solution of the Domino Problem, which is to find an algorithm that may decide whether any given Wang tile set admits tilings or not.

An astonishing answer to both the Domino Problem and the existence of a non-periodic Wang tile set was given by Wang's student Robert Berger. He showed that a set of Wang tiles can be translated into a Turing machine that does not halt if and only if the Wang tiles admit tilings. Undecidability of the Domino Problem then follows from undecidability of the Halting Problem. In proving this, Berger found the first aperiodic tile set. This notoriously consisted of a whopping 20,426 Wang tiles [2], although his original thesis already contains a smaller set of (after a minor correction) 103 tiles [4].

Soon after Berger's discovery, smaller aperiodic tile sets were discovered. Raphael Robinson found a set of just six tiles (although rotation and reflection of these is also allowed), given by squares with dents, bumps and other decorations required to be carried over tile edges, which forces a hierarchy of arbitrarily large square markings. Recently, Jeandel and Rao found an aperiodic Wang tile set of just 11 tiles [4], with edges decorated by 4 colours, and showed with the assistance of a computer search that there is no aperiodic Wang tile set with fewer tiles.

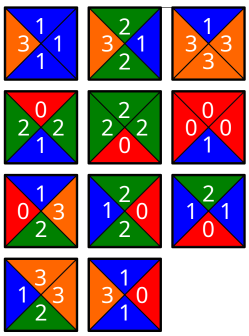

Figure 5. The aperiodic set of 11 Wang tiles, discovered by Jeandel and Rao; translated copies must be placed edge to edge with matching colours (equivalently matching numbers)

Dropping the requirement for tiles to be based on squares allows for even smaller tile sets, the most famous being Penrose's set of just two tiles. These are represented either with a pair of thin and thick rhomb (Figure 4), or as kite and dart shaped tiles (either choice results in tilings which are MLD). The edges are also decorated (alternatively given bumps and dents) which restrict how tiles may be placed adjacent to each other. Although uncountably many different tilings may be made from these two tiles (which can be translated or rotated by multiples of $2\pi/10$ radians), all such are non-periodic and locally isomorphic, meaning that any finite patch of one can be found in any other, up to translation.

Naturally, one wonders if there exists a tile set of just a single tile, a so-called 'einstein' (in German: *one stone*), admitting only non-periodic tilings of the plane using congruent copies of the tile. Amateur mathematician Joan Taylor found a candidate, developing the idea with Joshua Socolar [10]. The Socolar–Taylor tile may be represented either as a tile with disconnected interior or as a decorated hexagonal tile, but where matching rules require tiles to communicate not just between touching tiles but also next nearest neighbours. Similarly to the Robinson tiling, the matching rules force a hierarchy, this time of triangle decorations analogous to the construction of the Sierpiski gasket. It is still an open question as to whether or not there is a single tile which is a topological disc and only tiles non-periodically, with jigsaw puzzle or meeting colour types of matching rules.

Upside down fractals

If you want to open an aperiodic tiling factory, you probably do not want to focus production on matching rule tilings. Finding them is hard: Berger's result shows that it is generally undecidable to determine if a given tile set tiles space. Even if given such a tile set, it can be difficult to determine the global properties of the forced tilings, as the matching rules are local. But there is no reason to limit consideration to those tilings where aperiodicity is forced by the shapes of tiles.

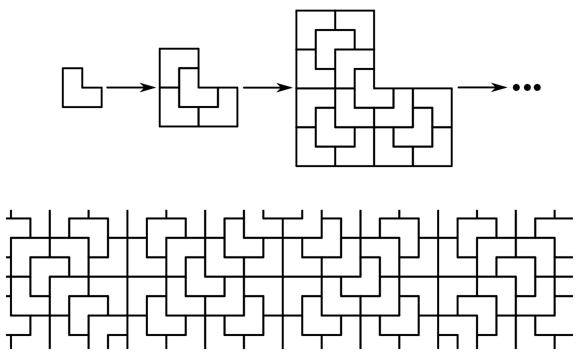

Figure 6. The chair substitution σ and patch of an infinite tiling admitted by σ; here $\mathcal{P}$ consists of 4 tiles (four chairs, 90° rotations of each other) and the inflation factor is $\lambda = 2$

One construction method of many interesting aperiodic tilings is substitution. To define a substitution rule σ one begins with a finite set $\mathcal{P}$ of so-called 'prototiles', tiles in $\mathbb{R}^d$. For each $p \in \mathcal{P}$ one defines a patch $\sigma(p)$ of translated copies of prototiles in $\mathcal{P}$. The patch $\sigma(p)$ should in some sense be geometrically related to the inflated version λp of p, for a fixed inflation constant $\lambda > 1$ (more generally, one may replace the inflation $x \mapsto \lambda x$ with an expansive linear map). For example, many substitution rules, called stone inflations, have $\sigma(p)$ equal in support to λp. The chair substitution is an example (see Figure 6). The Penrose tilings also admit a substitution rule, with λ the golden ratio, although it is not a stone inflation: substituted tiles cover the support of their inflated prototiles and more. But substitution is consistent, in that repeated substitution only ever generates tiles which agree exactly, whenever they overlap interiors.

Substitution can then be iterated, defining larger and larger patches. This defines tilings 'in the limit'. More precisely, a tiling T by translated copies of prototiles in $\mathcal{P}$ is admitted by the substitution σ if every finite patch of T may be covered by a translated supertile $\sigma^n(p)$, for some $n \in \mathbb{N}$ and $p \in \mathcal{P}$. The set Ω of such tilings is called the hull. Under quite general conditions, it may be shown that $\Omega \neq \emptyset$, and for each $T \in \Omega$ there is a so-called supertiling $T' \in \Omega$, satisfying $\sigma(T') = T$. Geometrically, this means that tiles of T may be grouped into level-1 supertiles, and replacing them with inflated copies of the original prototiles gives $T' \in \Omega$, after rescaling. Often it is not hard to see that this can be done uniquely (the so-called recognisability property), in which case non-periodicity follows.

Thus, one may think of substitution tilings as 'upside down fractals'. For self-similar fractals, one sees the same structures at finer and finer scales as one zooms in. For a substitution tiling, after some redecorations (such as grouping of tiles) the same structures recur repeatedly as one zooms out. This sort of hierarchy can be seen quite clearly in the Robinson and Taylor–Socolar tilings, and indeed these can be generated by substitution rules too.

Generally, substitutions that are primitive (meaning there is some $n \in \mathbb{N}$ with $\sigma^n(p)$ containing a copy of each prototile, for all $p \in \mathcal{P}$), and generate FLC tilings, in fact generate tilings which are all locally isomorphic and linearly repetitive. This implies complexity $p(r) \asymp r^d$ (for d the ambient dimension) and patches recur with well-defined frequencies. Unlike for matching rules, we start with a tool which already directly encodes a lot about the long-range order.

Substitutions can be generalised in several ways, such as allowing prototiles to also be rotated: the non-translationally FLC pinwheel tilings have only a single tile type up to isometry (a 1–2–$\sqrt{5}$ triangle) but substitution generates tiles in infinitely many rotational orientations. Another generalisation is to S-adic substitutions, where a sequence of different substitutions are applied to generate patches. A wonderful treasury of substitution rules and tilings can be found at the University of Bielefeld's Tiling Encyclopedia https://tilings.math.uni-bielefeld.de/.

One-dimensional substitutions

Substitutions are even of interest in one dimension, where tiles are (labelled) intervals. If we forget the lengths of tiles, these may be considered purely symbolically.

For example, the Fibonacci substitution is defined by $a \mapsto ab$ and $b \mapsto a$. This may be iterated: $a \mapsto ab \mapsto aba \mapsto abaab \mapsto \cdots$. The bi-infinite Fibonacci words thus generated are non-periodic and Sturmian, meaning there are $n+1$ words of any length $n \in \mathbb{N}$ (the minimal complexity a bi-infinite sequence can have without being periodic). To obtain a geometric substitution, a tiles can be replaced by intervals of length φ and bs of length 1. Then φa is exactly covered by ab (since $\varphi^2 = \varphi + 1$) and φb is exactly covered by a.

When introducing substitution tilings, we noted that there was no need to restrict to examples forced by matching rules. However, a deep result of Goodmann-Strauss shows that substitution tilings are always locally derivable from matching rule tilings [3]. This is a startling result: self-similar hierarchical order can always be forced by local rules.

Irrational slices

If you want a worthy rival to Substitution Corp's highly ordered aperiodic patterns, you should set up a cut and project factory. The idea is to start with something periodic (and thus ordered), a lattice, but to then break its periodicity, whilst keeping some of its order, by taking an irrational slice of it.

For a Euclidean cut and project scheme, one begins with a lattice Γ in the *total space* $\mathbb{E} \cong \mathbb{R}^k$. We then choose complementary subspaces $\mathbb{E}_\vee$ and $\mathbb{E}_<$, of dimensions $d \geq 1$ and $n \geq 1$, respectively. The subspace $\mathbb{E}_\vee$ is called the physical space, and is where the final pattern will live; $\mathbb{E}_<$ is called the internal space. The notation has been chosen to make things easier to remember in a first exposition: $\mathbb{E}_\vee$ is the subspace we project *down* to, and $\mathbb{E}_<$ is the subspace we project *left* to, at least in the 2-to-1 case ($k = 2$, $d = 1$) when we are looking at the first quadrant (see Figure 7). The projection to the physical space, parallel to the internal space, is denoted $\pi_\vee : \mathbb{E} \to \mathbb{E}_\vee$, and for short we write $x_\vee := \pi_\vee(x)$ (and analogously for the internal space).

Next we choose a window $W \subset \mathbb{E}_<$. We slide this parallel to the physical space to obtain the strip $\mathcal{S} = \mathbb{E}_\vee + W$. We then *cut* the lattice, by keeping only those points $\Lambda \cap \mathcal{S}$ in the strip, and *project* those to the physical space. The finished article is a point set $\Lambda := (\Gamma \cap \mathcal{S})_\vee \subset \mathbb{E}_\vee$, called a cut and project set.

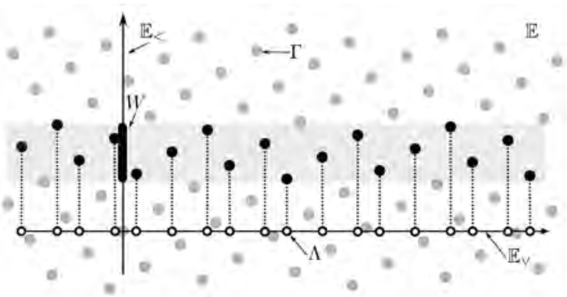

Figure 7. A 2-to-1 cut and project scheme

To get a nice Delone set, W is typically taken as compact and equal to the closure of its interior, such as a polytope, although fractals are also often used, a notable example being the Rauzy fractal. For canonical cut and project sets, W is taken as the projection to $\mathbb{E}_<$ of the unit hypercube, and $\Gamma = \mathbb{Z}^k$ in $\mathbb{E} = \mathbb{R}^k$. To obtain a non-periodic pattern, we ask that $\pi_<$ is injective on Γ. It is also usual to ask that $\pi_\vee$ is injective on Γ, so that a cut and project set may be identified with its lift to the total space. Finally, it is useful in their analysis for $\Gamma_<$ to be dense in the internal space.

A cut and project scheme, as above, defines an infinite family of non-periodic Delone sets, by first translating the lattice Γ, then cutting and projecting it. When the above restrictions hold, the resulting patterns are repetitive, so long as we use a non-singular translate of Γ, meaning it does not intersect the boundary of the strip (for singular translates, one needs to be careful on which points on the boundary of the strip to keep and remove).

Important examples are given by the vertices of the Ammann–Beenker tilings, generated by a canonical 4-to-2 cut and project scheme. These tilings are also generated by a substitution rule and exhibit 8-fold rotational symmetry. Vertices of the Penrose rhomb tilings are also cut and project sets. In this case, they are given by a canonical 5-to-2 projection, although this description is deficient in that it does not satisfy the above properties. It can instead be constructed using a 4-to-2 cut and project scheme, based on the A_4 root lattice, with internal space $\mathbb{R}^2 \times (\mathbb{Z}/5\mathbb{Z})$ and window a union of 4 pentagons in the internal space. Here we see that, more generally, non-Euclidean schemes can be used. For example, the chair tilings (up to MLD) may be constructed by a cut and project scheme with two-dimensional 2-adic internal space $\mathbb{Q}_2 \times \mathbb{Q}_2$.

The cut and project method is interesting for several reasons. Firstly, it constructs aperiodically ordered Delone sets with pure point diffraction, provided the window is sufficiently regular (such as having measure zero boundary [9]). Secondly, as for substitutions, the construction itself provides a tool with which to study the long-range order of the produced patterns. Thirdly, this construction has a number-theoretic flavour. It can be shown that a certain class of Euclidean cut and project schemes with polytopal windows are linearly repetitive if and only if they have complexity $p(r) \ll r^d$ (which itself may be determined algebraically from the data), and the scheme is Diophantine [5]. Loosely speaking, this means that the physical space does not get close to the lattice, relative to a lattice point's distance from the origin. For 2-to-1 schemes, and $\Gamma = \mathbb{Z}^2$, this is equivalent to the slope of the physical space being badly approximable by rationals, which in turn is equivalent to it having continued fraction expansion with bounded entries. For example, the one-dimensional Fibonacci tilings may be constructed from a 2-to-1 cut and project scheme whose physical space has slope the golden ratio $\varphi = 1 + \frac{1}{1+\frac{1}{1+\cdots}}$. Canonical 2–to–1 cut and project sets may always be generated by S-adic substitutions, where there are two substitutions acting in an order determined by the continued fraction of the slope of the physical space.

The above is consistent with the Fibonacci tilings being linearly repetitive as generated by substitution.

But knowing they are cut and project sets also implies they are are pure point diffractive, as well as giving a tool to compute the diffraction measure i.e., the locations of the Bragg peaks and their intensities.

The same holds for the Penrose tilings, which may be seen as 2-dimensional cousins of the 1-dimensional Fibonacci tilings (in fact, Fibonacci sequences can be found in Penrose tilings as 'Conway worms'). From their repeated appearance we see what makes the Penrose tilings such magical and prototypical examples in the field: they are forced by matching rules on a prototile set of just two tiles (up to rigid motion); they are linearly repetitive, since they may be defined by a primitive substitution rule; they have 5-fold (and 10-fold statistical) symmetry and this symmetry may be observed through their diffraction pattern (similar in appearance to that of Figure 2) which may be calculated using a cut and project scheme. The Penrose tilings do it all!

The mathematical theory of Aperiodic Order

This article has mostly concentrated on what Aperiodic Order is and how to construct examples of it. We finish by considering how mathematicians can approach their study.

The short answer is that almost every field of Mathematics seems to be valuable in Aperiodic Order! We have already seen aspects of Discrete Geometry, Fourier Analysis, Computational Logic and Diophantine Approximation. Aperiodic patterns define more abstract mathematical objects, such as inverse semigroups, topological groupoids and C^*-algebras, opening up their study to the tools of Noncommutative Geometry. The K-theory of these C^*-algebras is of physical relevance to quasicrystals, through Bellissard's Gap Labelling Theorem on the spectral gaps of Shrödinger operators on aperiodic patterns.

Many of these ideas meet in the topological and dynamical analysis of Aperiodic Order, which we now focus on (more details can be found in the excellent introduction [8] to this subject). For any non-periodic but FLC and repetitive aperiodic tiling T (such as a Penrose tiling), there are uncountably many locally isomorphic tilings. For substitution tilings, these are the tilings of the hull Ω mentioned before. For cut and project sets the family is given by the non-singular cut and project sets, along with singular limits of these. Generally, we define the hull (or tiling space) of T as:

$$\Omega = \{\text{tilings locally indistinguishable from } T\},$$

where T' is locally indistinguishable from T if every finite patch of T' can be found in T, up to translation.

There's typically no reason to study one tiling of the hull over any other (espeically in the repetitive case, when they are all locally isomorphic), so it is natural to consider the entire collection Ω together. This set has a natural (metrisable) topology, where two tilings are considered to be 'close' if they agree to a large radius about the origin, perhaps after translating either tiling by a small amount (this topology typically needs a mild but obvious modification in the non-FLC case, such as also allowing small rotations for the pinwheel tilings). Translation by $\mathbb{R}^d$ of tilings acts continuously, making $(\Omega, \mathbb{R}^d)$ a topological dynamical system.

Examples of some simple hulls

Suppose that T is fully periodic in $\mathbb{R}^d$, meaning it is invariant under translation by a full rank lattice $\Gamma \leqslant \mathbb{R}^d$, so that $\Gamma \cong \mathbb{Z}^d$. It is not too hard to see that T' is locally indistinguishable from T if and only if T' is a translation of it, so the hull is homeomorphic to the d-torus $\Omega \cong \mathbb{R}^d/\Gamma$.

For a non-periodic example, consider the tiling X of $\mathbb{R}$ of unit interval tiles, except with one tile of length 2. All translates of X are locally indistinguishable from it, but so are translates of the fully periodic tilings of unit intervals. These form a subspace of the hull homeomorphic to a circle, and the 'leaf' containing X winds towards it as X is translated to the left or right.

Now take the tiling of $\mathbb{R}$ whose interval tiles are eventually all length 2 on the left and eventually all length 1 on the right. Why does the hull look like a Helical Stair Traversing Toy$^{\text{TM}}$?

The hull of an FLC tiling is always compact and connected, but will not be path-connected for a non-periodic tiling. Its path components are in bijective correspondence with translational orbits of tilings in Ω. And it is a fibre bundle (a kind of twisted product) over a d-torus, with totally disconnected fibre, a Cantor set for non-periodic but repetitive tilings.

One might imagine it would be difficult to say much more about such a pathological space. In fact, topological invariants of Ω (and thus MLD invariants of the original tiling) can often be calculated when the hull is generated by substitution or the cut and project method. For example, the Cech cohomology groups of the Penrose tilings may be calculated as $\mathbb{Z}$, $\mathbb{Z}^5$, $\mathbb{Z}^8$ in degrees 0, 1 and 2, respectively, and are trivial in other degrees. Invariants like this not only distinguish between tilings, but more importantly store useful information about them.

The topological dynamics store further geometric and statistical information. The tiling has well-defined patch frequencies if and only if $(\Omega, \mathbb{R}^d)$ is uniquely ergodic (that is, it has a unique invariant measure). Even within the generalised setting of translation bounded measures, pure point diffraction is equivalent to $(\Omega, \mathbb{R}^d)$ having pure point dynamical spectrum, which is a conjugacy invariant of the dynamical system. In many cases of interest, including FLC, primitive substitution tilings, this happens if and only if Ω factors almost everywhere 1-to-1 to its so-called maximal equicontinuous factor. Even outside of the pure point case, there is a certain correspondence between the dynamical and diffraction spectrum of elements of topological factors of Ω.

Naturally, we have concentrated on very special tilings above, constructable both by substitution and cut and project schemes. But codimension $n \geq 2$ cut and project sets generically have rational Cech cohomology groups with infinite rank, which never happens for FLC substitution tilings. Conversely, not all substitution tilings are cut and project sets with reasonable window, as not all substitution tilings have pure point diffraction. The question of which substitutions have pure point diffraction, or can be represented with cut and project sets with nice windows, is a deep problem. In dimension one we have the Pisot Conjecture in this direction, which although solved for some special cases remains open in full generality.

A 'mathematical invitation' to the field can be found in Michael Baake and Uwe Grimm's book [1] (the first of two current volumes). Tragically, Uwe unexpectedly died in October 2021, aged 58, a great loss still felt acutely by everyone who knew him. A wonderful meeting (see page 12 in this issue) in his honour was held recently at the Open University, combining research talks on Aperiodic Order and an exhibition of art (both visual and musical), mathematical sculptures and interactive tiling puzzles, which perfectly combined his research passions and engagement in public outreach.

James J. Walton

Jamie is a Teaching Associate at The University of Nottingham. His main mathematical research interests are in the intersection of Aperiodic Order with Algebraic Topology, Dynamical Systems and Diophantine Approximation. When not cutting or projecting, Jamie enjoys long walks, baking bread and playing classical guitar. He also likes board games, when permitted by resident voids Isaac (pictured) and his sister Kiki.

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How to win Nim games and code them in LuaLaTeX

PETER ROWLETT

Abstract. Solving the game Nim, writing a LaTeX package to typeset it, and programming game solutions.

Introduction

The first thing I did when I heard we had approval to run a new final year module 'Game Theory and Recreational Mathematics' was buy a thousand matches. I was sure I would want to include Nim in the game theory section of the new module. It was not so easy to work out how to typeset Nim games in my lecture notes, which ultimately led to me writing a new LaTeX package. Later, when I was playing with programming in LuaLaTeX, solving multi-pile Nim games seemed an interesting challenge.

This article discusses Nim and its solution, outlines typesetting games in LaTeX and uses this as an example for an introduction to LuaLaTeX programming.

Nim

Nim is an ancient game, that may have originated in China and has been in Europe since at least the 16th century. The name is more recent, being attributed to Charles L. Bouton, who published a quite readable mathematical analysis of the game in 1901 [1].

Nim is played with a number of piles (or heaps) of objects, usually sticks, stones or counters. Two players alternate moves. A move is to remove a positive number of sticks from precisely one heap. The aim is to take the last stick.

For example, Alice and Bob play a game with three heaps of two, three and four sticks.

|\ ||\   /|\/

Moving first, Alice reduces the heap of four to one.

//   \|\   /

Unsure what to do, Bob removes the heap of three.

|/   \

Figure 1. Match sticks. Photo by Katie Steckles

Nim as a combinatorial game

Nim meets the following conditions for being considered a combinatorial game:

- it is *two-player*;
- it is *turn-based*: the two players alternate turns throughout the game;
- it is *deterministic*: there is no chance;
- players have *perfect information*: each player is aware of the details of the game position and the set of moves available to both players at all times;
- it is *finite*: there are a finite number of positions that can be taken;
- it *ends*: the game will end in a finite sequence of moves;
- there is a *winner*, which is typically the last player to move for *normal play* (the last player to move loses in *misère play*).

Nim is an *impartial game*, in that the available options are the same for both players. We will see that Nim is a *solved* game, in that the outcome can be correctly predicted from any position assuming *perfect play* (each player playing optimally in their own best interest).

Alice takes one stick from the heap of two.

Realising he's beaten, Bob removes one stick and leaves Alice to take the other.

Nim sum

The key to analysing Nim positions is a concept called Nim sum, denoted $\oplus$. We convert the sizes of the heaps into binary and then the Nim sum is the bitwise XOR addition of the heap sizes. For example, the opening position of Alice and Bob's game with heaps of two, three and four sticks has Nim sum

	0	1	0
	0	1	1
$\oplus$	1	0	0
	1	0	1

Nim-sum is commutative and associative with $a \oplus 0 = a$, and satisfies $a \oplus b = 0$ if and only if $a = b$.

Let $X = x_1 \oplus \ldots \oplus x_n$ be the sum of the sizes of n heaps in a multi-pile Nim game, and let $Y = y_1 \oplus \ldots \oplus y_n$ be the sum of the sizes of those heaps after a move has been made.

Since at the end of the game no sticks remain, this position has Nim sum zero. We can prove that you can make $Y = 0$ if and only if $X \neq 0$. This means that if you can get the game into a Nim sum zero position, your opponent must change it to a non-zero position and you can keep returning it to a zero position. Since the sizes of the heaps strictly decrease, you obtain the winning position, taking the last stick, in a finite number of moves.

Making a move can only change one pile by decreasing its size. Say the move was in heap k. Then $x_i = y_i$, i.e. $x_i \oplus y_i = 0$ for all $i \neq k$ and $x_k > y_k$, i.e. $x_k \oplus y_k \neq 0$.

Consider

$$\begin{aligned} X \oplus Y &= (x_1 \oplus y_1) \oplus \ldots \oplus (x_n \oplus y_n) \\ &= 0 \oplus \ldots \oplus (x_k \oplus y_k) \oplus \ldots \oplus 0 \\ &= x_k \oplus y_k. \end{aligned}$$

We know that $X \oplus Y \neq 0$ i.e. $X \neq Y$ because $x_k \oplus y_k \neq 0$. This means that if the game is in Nim sum zero position, $X = 0$, you must leave it in a non-zero position, $Y \neq 0$, after your move.

The fact that $X \neq Y$ offers the possibility of changing a non-zero position $X \neq 0$ to a zero one $Y = 0$.

Now take

$$X \oplus (x_k \oplus y_k) = X \oplus (X \oplus Y) = Y.$$

If $X \neq 0$ then there is at least one non-zero bit in X. Say the most significant non-zero bit occurs in the dth place. Then there must be an x_i with a 1 in the dth place also. Choose this as x_k. Then choose $y_k = x_k \oplus X$ so that

$$\begin{aligned} Y &= X \oplus (x_k \oplus y_k) \\ &= X \oplus (x_k \oplus (x_k \oplus X)) \\ &= (X \oplus X) \oplus (x_k \oplus x_k) \\ &= 0 \oplus 0 = 0. \end{aligned}$$

This means it is always possible to convert a non-zero Nim sum position to a zero one by careful choice of x_k and y_k.

In the game with heaps 2, 3 and 4, Alice calculated $X = 2 \oplus 3 \oplus 4 = 5$. Since 4 (100) and 5 (101) share a leading bit 1, Alice chose the 4-heap as x_k. She changed this to $y_k = x_k \oplus X = 4 \oplus 5 = 1$, resulting in $Y = 2 \oplus 3 \oplus 1 = 0$. At this point Bob could not win. Each move, he made the Nim sum non-zero and Alice returned it to zero until there were no sticks left.

Winning a combinatorial game

For a combinatorial game that cannot end in a draw played between Alice and Bob, either Alice can force a win moving first, or Bob can force a win moving second, but not both.

A game position from which the next player to play can force a win is called an $\mathcal{N}$-position.

A game in $\mathcal{P}$-position is one where the previous player who played (or the second to play, since play alternates) can force a win.

So a Nim game position with Nim sum zero is a $\mathcal{P}$-position, whereas a position with non-zero Nim sum is an $\mathcal{N}$-position.

nimsticks

I write notes in LaTeX. There are various types of dot in LaTeX that might look like stones or lines, but for me with my messy piles of matchsticks I wanted something that looked thrown together and options like '• • •' or '| | | | |' just looked too regular. Really, I wanted a set of lines that look like matchsticks someone just threw into heaps, though without crossings for the avoidance of ambiguity.

TikZ is a package for drawing in LaTeX. It is straightforward to draw a line in TikZ:

```
\begin{tikzpicture}
    \draw[very thick] (0,0) -- (0,0.5);
\end{tikzpicture}
```

This produces a straight vertical line like this:

I wrote a command `\drawnimstick` which generates two random numbers in $[-0.1, 0.1]$ and adds these to the top and bottom of the stick to give it a little wobble, like this:

Then I wrote a command `\nimgame{}` that takes heap sizes and draws wobbly lines for those heaps. For example, `\nimgame{3,4,5}` produces

For a while, I just had a nice bit of code. One lazy Sunday morning I took the pluge and learned how to submit this as a LaTeX package called nimsticks.

A LaTeX package is made of a sty file containing the commands, a PDF of documentation explaining how to use the package and a README file containing a basic description of the package.

It's quite common to actually make a dtx file and an ins file, and then these generate the sty and PDF files. This is a bit fiddly; I followed a guide by Matthew Scroggs [2]. If you have a useful bit of LaTeX code written up, I'd encourage you to submit it as a package.

If you make a LaTeX document with `\usepackage{nimsticks}` you can use `\nimgame` to draw games.

nimsticks on the web

You can view the Comprehensive TeX Archive Network page at ctan.org/pkg/nimsticks.

The source code is available via github.com/prowlett/nimsticks

If you spot any bugs or other issues there is an issues page on GitHub where you can post these.

LuaLaTeX

TeX was created by Donald Knuth and LaTeX was built on top of TeX by Leslie Lamport. Originally the output was DVI/Postscript, but later pdftex and pdflatex provided PDF output.

There are a couple of systems trying to modernise TeX, including XeTeX, which aims to support a wider range of characters and fonts, and LuaLaTeX, which aims to act like LaTeX with a more user-friendly programming language, Lua, built in.

You may be able to compile your LaTeX file using lualatex instead of pdflatex, though some packages don't work with LuaLaTeX. In fact, when getting my nimsticks package together I changed the way it generates random numbers so it would work with pdflatex, xetex and lualatex all.

Within a LuaLaTeX document you can write Lua code using a package luacode and its environment `luacode`. For example, here is a simple LuaLaTeX document that uses a for loop to print powers of x from x^2 to x^{10}. Note that Lua uses \ so in the command `tex.sprint()` which outputs to TeX we use \\ where we want TeX to see \.

```
\documentclass{article}
\usepackage{luacode}
\begin{document}
A list of powers:
\begin{itemize}
\begin{luacode}
for i=2,10 do
  tex.sprint("\\item \\( x^{",i,"}\\)")
end
\end{luacode}
\end{itemize}
\end{document}
```

Why is this a good thing? Well, Lua is quite a straightforward and useable programming language and generally less fiddly to use than programming in TeX directly. It has a math library which contains lots of useful functions.

As an example, if we want to typeset a random three pile Nim game, we can do this.

First we need to generate the sizes of the three heaps. All the code goes within a `luacode` environment. Here the Lua command `math.random` is used within a for loop to generate random numbers from 3 to 7. These are stored within an array called `heaps`.

```
heaps={}
for h=0,2 do
  heaps[h]=math.random(3,7)
end
```

Of course, here we have just generated our random values, we haven't actually told LuaLaTeX to display anything. Here we use `tex.sprint()` to output to LaTeX a `\nimgame` command with the three generated values. This will display a randomly-generated Nim game.

```
tex.sprint("\\nimgame{", heaps[0], ",",
heaps[1], ",", heaps[2], "}")
```

This will produce a LaTeX command like `\nimgame{5,5,6}` with our randomised heap sizes. LaTeX will interpret this as a call to the nimsticks package and produce the appropriate graphic.

The first step to analysing a Nim position was to convert the heap sizes to binary. Here we write a function that stores the binary bits in an array. Since our heap sizes are between 3 and 7 we require 3 bits. The for loop counts backwards over `b` from 2 to 0 so we consider the least significant bit first. In each iteration we compute the remainder of dividing the heap size by 2 using `math.fmod()`, store this in `bits` and then reduce the remaining heap size.

```
function nimtobin(heap)
  bits={}
  for b=2,0,-1 do
    rem=math.fmod(heap,2)
    bits[b]=rem
    heap=(heap-rem)/2
  end
  return bits
end
```

For example, consider 6. The remainder of dividing $6/2$ is 0, so we store this as the third bit (with index `b`=2). We reduce 6 to $(6-0)/2=3$. Next iteration, the remainder of $3/2$ is 1, so we store this as the second bit. Then we reduce 3 to $(3-1)/2=1$. Third iteration, the remainder of $1/2$ is 1, so we store this as the first bit. Reducing 1 to $(1-1)/2=0$ completes the loop. The values stored in `bits` are 1, 1 and 0, representing the binary number 110.

As we have a function to convert our decimal heap sizes to binary, it will be useful to be able to convert these back to decimal. Here `math.ceil()` is just used because the output of the calculation is a float (e.g. `1.0`) and it looks neater to convert this to an integer.

```
function bintodec(heap)
  return math.ceil(4*heap[0]+2*heap[1]
  +heap[2])
end
```

Above, we generated three heaps with decimal heap sizes. We wish to work with these as binary bits, so we convert these.

```
bitheaps={}
for h=0,2 do
  bitheaps[h]=nimtobin(heaps[h])
end
```

Now we have our three heaps in binary we require a function to calculate their Nim sum. Here we loop over the three bit places. For each, we sum the values of the corresponding bit in the three heaps, then compute the remainder of dividing this sum by 2. The command `#heaps` returns the highest index used in `heaps`. It is used here instead of 2 because we will see later that we want to compute the Nim sum of two numbers instead of the three here.

```
function nimsum(heaps)
  bits={}
  for b=2,0,-1 do
    this=0
    for h=0,#heaps do
      this=this+heaps[h][b]
    end
    bits[b]=math.fmod(this,2)
  end
  return bits
end
```

We call our new function to compute X, the Nim sum of our binary heap sizes.

```
X=nimsum(bitheaps)
```

Recall that if $X = 0$, there is no winning move.

```
if bintodec(X)==0 then
  tex.print("Nim sum is zero.")
end
```

If $X \neq 0$, a winning move can be found. The remaining code would be within an `else` on the above `if` statement.

First we identify the position of the most significant bit of X, referred to earlier as the dth place.

```
for b=2,0,-1 do
  if X[b]==1 then
    d=b
  end
end
```

Next we select x_k by identifying any pile that has 1 in the same position as the most significant bit of X. This code will find the last heap with a 1 in the dth place.

```
for h=0,2 do
  if bitheaps[h][d]==1 then
    xk=bitheaps[h]
  end
end
```

Finally, we compute $y_k = x_k \oplus X$ using our `nimsum()` function again.

```
sumparts={}
sumparts[0]=xk
sumparts[1]=X
yk=nimsum(sumparts)
```

One last command outputs the winning move.

```
tex.sprint("Change ",bintodec(xk),
" to ",bintodec(yk))
```

Putting our code in a loop could be used to generate questions and answers for a worksheet, for example.

You can view the completed program from this example at github.com/prowlett/lualatex-nim.

The result

The output of our program is something like:

Change 4 to 2

Here the Nim sum is

$$X = 7 \oplus 5 \oplus 4 = 111 \oplus 101 \oplus 100 = 110.$$

Here any pile is a candidate for x_k as they all have a leading bit 1. The code has selected the 4-heap and has computed $y_k = x_k \oplus X = 100 \oplus 110 = 010$.

Conclusion

We have seen how to include Lua code in a LuaLaTeX document and explored some basics including for loops, if statements and printing, along with some mathematical functions. There is a lot more to LuaLaTeX, including some subtlety glossed over in this article. I would encourage you to have an explore and play at writing your own code.

For simplicity of code in this article, we relied on the limited number of heaps with limited range of sizes. These restrictions could be removed, and it may be an interesting exercise for the reader to write a version with either a random number of heaps or an arbitrary range of heap sizes.

FURTHER READING

[1] C. L. Bouton, Nim, A Game with a Complete Mathematical Theory, Ann. Math. 3 (1901) 35–39.
[2] M. Scroggs, realhats: Writing a LaTeX Package, *The Aperiodical*, 2019, tinyurl.com/realhatsarticle.

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Notes of a Numerical Analyst

Nonsmooth Landscapes

NICK TREFETHEN FRS

Scrambling through the brush on a Welsh hillside last summer, Kate and I ran into a problem. You might expect the ground to be dry high up, but here in the ravine between two little slopes, the water was a foot or two deep! How could we get across without soaking our boots?

Somehow we did, and for the next few miles, I found myself thinking about water and topography. I am sure most of us have considered maxima, minima and saddle points in outdoor landscapes, and these concepts make good sense if the surface $f(x,y)$ is smooth. What I hadn't noticed is that to understand creeks and streams, you need to let f be nonsmooth.

In the simplest mathematical idealization, infinitesimal rivulets flow down paths of steepest descent. If the surface is smooth, as in the left image of Figure 1, though the rivulets grow denser in the bottom of a valley, they remain distinct. This does not explain the water that troubled us on our hike. For that, you need a nonsmooth surface, as in the image on the right. Now, thanks to the singularity, multiple rivulets can coalesce. The flow is irreversible in the sense that in reversed time, the trajectories are nonunique.

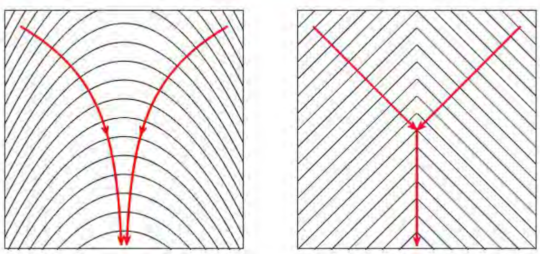

Figure 1. Contours of landscapes $f(x,y) = x^2 + y$ (left) and $|x| + y$ (right). The singularity on the right enables steepest descent paths to coalesce. Thus even a small ravine high on a hillside may have a good deal of water at the bottom.

When mathematicians investigate the world, we usually begin with smooth models, but sometimes nonsmoothness turns out to be essential. A familiar example is sonic booms from supersonic aircraft. Somehow, in a lifetime of hiking, I had not noticed that creeks and streams are another example.

Figure 2. It is rather a long way from Figure 1 to what we see outdoors. (Photo from iStock.)

Whenever something is nonsmooth, there is the question of how it got that way—often as a result of nonlinearity. In the case of landscapes and rivers, an obvious nonlinearity is that flowing water carves and changes the surface. The study of river channellization is a well developed topic of geomorphology, towards which Figure 1 represents only the tiniest of steps [1, 2]. I am grateful to Greg Tucker of the University of Colorado for teaching me about this subject.

FURTHER READING

[1] A. Fowler, *Mathematical Geoscience*, Springer 2011.
[2] I. Rodríguez-Iturbe and A. Rinaldo, *Fractal River Basins*, Cambridge, 1997.

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Mathematics News Flash

Jonathan Fraser reports on some recent breakthroughs in mathematics.

A cone restriction estimate using polynomial partitioning

AUTHORS: Yumeng Ou, Hong Wang
ACCESS: https://arxiv.org/abs/1704.05485

The *Fourier restriction conjecture* is a fundamental problem in harmonic analysis. It asks whether the Fourier transform of a function can be meaningfully restricted to a hypersurface in the sense that the L^p norm of the restriction of the Fourier transform can be controlled by the L^p norm of the function for some range of p. For this to be realistic, some assumptions on the hypersurface are needed and Stein famously made explicit conjectures involving curvature features of the hypersurface.

The truncated cone (in arbitrary ambient dimensions $n \geq 3$) is a key example: it exhibits curvature but the curvature vanishes in certain directions. Stein explicitly conjectured that the restriction conjecture should be true for the cone for $1 \leq p < 2(n-1)/n$. Stein's conjecture was previously known to be true for $n = 3, 4$. This paper, published in *Journal of the European Mathematical Society* in 2022, proves the conjecture for $n = 5$ and improves the range of p for larger n. The approach is in part based on 'polynomial partitioning'. This is a powerful tool in harmonic analysis stemming from work of Guth and Katz which solved the Erdős distinct distances problem. This work, in turn, took inspiration from Dvir's famous resolution of the finite fields Kakeya problem.

Interpolation for Brill–Noether curves

AUTHORS: Eric Larson, Isabel Vogt
ACCESS: https://arxiv.org/abs/2201.09445

The *interpolation problem* is a fundamental problem in geometry which asks, roughly speaking, how many distinct points are needed to uniquely determine a curve in a given class. Fundamental examples include the (trivial) fact that two points uniquely determine a line and the (slightly less trivial) fact that three non-collinear points uniquely determine a circle. Another example — perhaps less familiar to the reader but still dating back to Pappus 340 AD — is that five points uniquely determine a conic section in the plane (subject to some technical geometric constraints somewhat similar to the non-collinearity required in the previous example).

Following much investigation in classical geometry, this problem now lives in modern algebraic geometry. Even understanding what an answer to the problem might look like is already fascinating with several contributions across many years. Central to the story is the Brill-Noether theorem (proposed in the 1870s and finally proved by Griffiths and Harris in 1980), which provides the most natural classes of curves to analyse. This celebrated paper, appearing on arXiv this year, fully resolves the interpolation problem.

On the implosion of a 3D compressible fluid

AUTHORS: Frank Merle, Pierre Raphael, Igor Rodnianski, Jeremie Szeftel
ACCESS: https://arxiv.org/abs/1912.11009

This substantial work, which is several hundred pages long and split over two papers published in 2022 in the same issue of *Annals of Mathematics*, constructs a set of finite energy smooth initial data for which the corresponding solutions to the compressible three dimensional Navier-Stokes and Euler equations implode. The authors state that their main goal was to construct a family of smooth, global in space, self-similar profiles which come from smooth initial data and proceed to blow up. Existence of self-similar solutions with spherical symmetry for the Euler equation with a continuum of admissible blow up speeds was already known via seminal work of Guderley and Sedov. However, the previously known solutions are either non-global or non-smooth.

Jonathan Fraser is a Professor at the University of St Andrews and an Editor of this Newsletter. He is pictured here on St Andrews pier with his son, Reuben. The photo was taken during a walk on Reuben's first birthday.

Research and Development in University Mathematics Education: Overview produced by the International Network for Didactic Research in University Mathematics

edited by Viviane Durand-Guerrier, Reinhard Hochmuth, Elena Nardi, & Carl Winsløw, Routledge, 2021, £32.99, ISBN: 978-0-367-36537-0

Review by Gabriel Chun-Yeung Lee

In the last thirty years or so, the need to address the issues of learning and teaching university mathematics has become increasingly appreciated by university teachers and researchers around the world. Whilst early research on the learning and teaching of mathematics at university can be seen in [1], this book, *Research and Development in University Mathematics Education* edited by Viviane Durand-Guerrier, Reinhard Hoshmuth, Elena Nardi and Carl Winsløw, sheds light on recent research activities and findings on university mathematics education, reported at two INDRUM (International Network for Didactic Research in University Mathematics) conferences organised in 2016 and 2018. Through synthesising the research that appeared at the two INDRUM conferences, this book highlights key research perspectives, addresses seminal theoretical and methodological issues and reports substantial results about the learning and teaching of mathematics at university level.

This book consists of 12 chapters covering three main themes, namely: (1) achievements and current challenges in university mathematics (Chapters 1–4), (2) learning and teaching of specific topics in university mathematics (Chapters 5–9), and (3) teachers' and students' practices at university level (Chapters 10–12). Each chapter, developed by different authors, reviews existing research on a specific topic within one of the three main themes, particularly contributions that were reported at the two INDRUM conferences. Whilst each chapter can be seen as an individual contribution of its authors with a specific focus, the chapters are related. For example, Trigueros *et al.* discuss the variety of research on the learning and teaching of calculus and analysis in Chapter 5, and Vandebrouck *et al.* discuss issues about task design in calculus and analysis in Chapter 6.

Apart from discussing the challenges that university students, including mathematics students and students of other subjects, often encounter in the learning and teaching of university mathematics (e.g., transitions from school mathematics to university mathematics), this book also discusses the issues related to university mathematics teachers — which particularly draw my attention. For example, in Chapter 4, Winsløw *et al.* focus on the common practices of university mathematics teacher education and the opportunities for university mathematics teacher development. After giving overviews of practices of university mathematics teacher education in Germany, Norway, the UK, and the USA, the authors draw on the findings to give four suggestions for improving the teaching of university mathematics, from personal development to institutional change.

Another idea that emerges from different chapters in this book is a need to develop stronger synergy between the communities of mathematics and mathematics education. It is believed that collaboration between the two communities could provide insights into addressing the multifaceted issues of teaching university mathematics and developing models of teaching university mathematics from the teaching of mathematics at other educational levels. I particularly like the question raised by Grenier-Boley *et al.* in Chapter 12: “are [the models of learning and teaching practices developed in school-based research] transferable to learning in higher education?” (p. 254) which can be transferred into a more general question “are the models of learning and teaching practices developed in one context transferable to learning in another context that is significantly different?” Of course, this book does not give an answer to the question and more research has to been done — I would love to learn more about the advancement of research concerning this question at future INDRUM conferences!

As a research synthesis of the first two INDRUM conferences, this book is generally written in a formal style — each chapter starts with an introduction which gives a description of different sections included in the chapter and ends with a closing discussion. Some chapters are written in an academic research paper style and some in a speech style, depending on the authors and the kind of research activities (e.g., a keynote lecture) the chapter was developed from. Overall this book is easy to read, and notes and references are available to readers for further reading in areas of educational research, aligned with the aims of INDRUM to support “the development of early-career researchers in the field” and to foster “dialogue with the mathematics community” (p. xv). It is impossible for me to do justice in such a short review to every part of this book, but I expect it to be a useful resource in research on university mathematics education to have on my bookshelf.

FURTHER READING

[1] Derek Holton, Michèle Artigue, Urs Kirchgräber, Joel Hillel, Mogens Niss and Alan Schoenfeld. The teaching and learning of mathematics at university: An ICMI study, Kluwer Academic Publishers, New York, 2001.

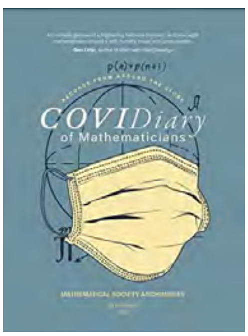
Gabriel Chun-Yeung Lee

Gabriel Lee is a DPhil student in the Department of Education, University of Oxford. His DPhil research topic is *Hong Kong preservice teachers’ beliefs and attitudes towards teaching proof in school mathematics: A design-based research*. His research interests covers school mathematics teacher education and the learning and teaching of proof.

COVIDiary of Mathematicians

by Aleksandra Ravas and Dragana Stošić Miljković (eds.), Mathematical Society Archimedes, Belgrade, 2021, ISBN: 978-8661750960

Review by Christopher D. Hollings



It is difficult to disagree with the publisher’s opening observation that this is an ‘unusual book’. It is the English translation of a volume that appeared in 2020 in a Serbian series that would normally consist of books of mathematical challenges [1]. As previously described in the *Newsletter* [2], however, this one takes the form of a diary of mathematicians’ experiences during the early stages (mostly April 2020) of the Covid-19 pandemic. True to the series’ usual format, some mathematical puzzles do appear (with detailed solutions and discussions at the end of the book). The volume is complemented by online resources (mostly videos) that are accessed via QR codes printed throughout.

The book emerged from the personal diary that the mathematician Tiago Hirth was invited to produce for a Serbian newspaper during the early days of the pandemic, and this appears as a prologue, chronicling life in Lisbon during April 2020. Each chapter that follows contains the diary entries of a different mathematician: Guido Ramellini in Barcelona; James Tanton in Phoenix, Arizona; Jovan Knežević in Belgrade; Kiran Bacche in Bangalore; Sergio Belmonte in Altafulla, Spain; and Tijana Marković, also in Belgrade. Editorial footnotes throughout help to explain aspects of local context.

As a whole, the book presents a range of occurrences and observations, in a variety of voices and styles, about what was happening in the world in April 2020, and yet it also contains an undercurrent of shared difficulties: local circumstances and the specific restrictions under which people lived may have differed in their details, but the experiences of the past couple of years, and the ways in which people have had to adapt, are seen to transcend borders.

Most of the experiences described in the book will be recognisable to anyone, not just mathematicians, and we see these emerging over the sucessive chapters: the sudden need to work from home, restrictions on leaving the house, the gradual introduction of health regulations, and so on. Some of the chapter authors do not spare their respective governments criticism over their response to the pandemic. Other themes that appear are: losing track of time, and throwing oneself into neglected hobbies. We are given a snapshot of the authors' daily activities, some of them mundane, but all of them showing how life continued. Adaptation to online life, both work and social, is addressed, as is the drawback of then spending the whole day in front of a computer screen. We read of the difficulties of online schooling, and the grim fascination of watching variations in Covid case numbers. Some of the authors describe venturing out into eerily deserted city streets.

Circumstances that are specific to mathematicians (or at least to academics) are also woven throughout the book: the tyranny of online meetings and their associated IT difficulties, the preparation and editing of lecture recordings, the question of electronic assessment, and the disparity of access to online teaching. But there are positives here as well: the way in which mathematics became very visibly useful during the pandemic, the broadening of access to mathematics that was brought about by the introduction of online events, and the solace that can be taken in interesting mathematics, recreational or otherwise.

The opening editorial remarks describe the volume as a 'miniature literary exercise', and it is perhaps with this in view that the book concludes with the reversed diary of Guido Ramellini, which runs backwards through April 2020. In contrast to Ramellini's earlier contribution to the book, this chapter is intended as a 'false diary': a series of musings for posterity, rather than an accurate record of events. And this concluding chapter features one of the most positive elements of the book overall: a description of the continuation of mathematics outreach activities even during a lockdown.

To return to the opening comment, this certainly is an unusual book, as mathematical books go, but it is all the better for it. It is nicely produced book, illustrated throughout by photographs and sketches, in many cases of and by the authors. It manages to be both a charming read and a valuable record.

FURTHER READING

[1] Aleksandra Ravas and Dragana Stošić Miljković (eds.), *KOVIDnevnik matematičara*, Matematichko Drushtvo Arhimedes, Beograd, 2020.
[2] COVIDiary, LMS Newsletter, no. 501 (July 2022), 9.

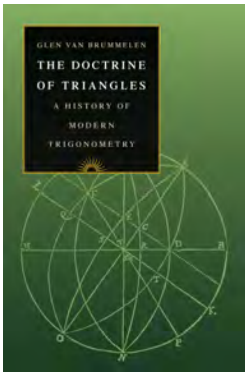
Christopher Hollings

Christopher Hollings is Departmental Lecturer in Mathematics and its History in the Oxford Mathematical Institute, and Clifford Norton Senior Research Fellow in the History of Mathematics at The Queen's College. His research covers various aspects of the mathematics of the nineteenth and twentieth centuries.

The Doctrine of Triangles: A History of Modern Trigonometry

by Glen Van Brummelen, Princeton University Press, 2021, £25, US$ 29.95, ISBN: 9780691179414

Review by Zhuo Zhang



Written as a sequel to his first volume on the history of trigonometry [1], Glen Van Brummelen's *The Doctrine of Triangles: A History of Modern Trigonometry* continues the same narrative. Trigonometry, although no longer an active area of research, gave birth to and inspired many modern disciplines, and it remains, today, fundamental to both science and trade. *The Doctrine* traces these lineages and provides a scientific recounting of their bloodlines throughout history. A significant amount of focus in this book is placed upon the European development of the subject, but Van Brummelen reaches far beyond the West in his work. Fulfilling a promise he pledged in the first volume, this book contains a 59-page chapter on trigonometry within China, from the early 3rd century CE to its assimilation into the Western canon in the late 19th century.

Aiding the reportorial style of this work are the frequent references to and the detailed analysis of the original materials. Van Brummelen set out to 'bring the existing research together' and 'return it to accessibility'. The clear transcription of source materials flanked by succinct diagrams laid out in modern notation sees to the fulfilling of those goals. However, despite his appeal to current methods for ease of access, Van Brummelen remains faithful to history and steers clear of modern biases on the subject. In addition to the section dedicated to Chinese trigonometry, he also draws special attention to areas of Western research often ignored by historians due to their tenuous link to our modern curriculum. One example of this is spherical geometry, which the book explores in detail.

Starting in the first chapter, *The Doctrine* picks up where the previous volume left off and starts the story in the 16th century. The protagonists here include figures such as Regiomontanus, Rheticus, Viète, Snell and Clavius and through them, we see how the concepts of basic trigonometric functions took shape, evolved, and were adapted into the toolboxes of mathematicians. Towards the end of this chapter, the book leads us into the applications of trigonometry in the physical world, such as surveying and navigation. This theme of trigonometry changing, evolving, and adapting to the ongoing development of mathematics remains pervasive throughout this work. And indeed, in Chapters 2 and 3, we see how the subject flourished under the new concepts of logarithms and calculus. In Chapter 5, Van Brummelen further explores the development of European trigonometry post-Euler, in areas such as stereographic projection and Fourier series.

In Chapter 4, Van Brummelen takes a detour from the history of European mathematics and delves into how trigonometry developed in China. The subject here presents itself as an interesting case study, as Chinese mathematics did not engage in extensive interactions with the rest of the world before the late 16th century, and it is in the exploration of this unfamiliar scene that the book shows its mastery over the subject matter. Each historical piece is treated with the respect it deserves using the methods employed by the original authors whilst the presentation is made accessible and palatable to the modern audience. Throughout the European chapters, the same dedication to historical accuracy could also be seen in the treatment of spherical geometry, which is a subject that enjoys relatively low popularity and recognition amongst modern mathematicians. Van Brummelen keeps the general reader's unfamiliarity with this area in mind, and in the relevant paragraphs, he provides analysis that proves to be approachable to most readers, who are also helped greatly by the accompanying diagrams.

Van Brummelen states that although he largely predicated his book on the scientific side of history, he has moved a little in the direction of social history compared to his previous volume. This step towards the latter can be seen throughout the book and provides a wonderful backdrop to mathematical paragraphs that might seem dry and unappealing by themselves. From the various quoted commentaries, dialogues, and even poems, we catch a brief glimpse of the mathematicians behind the work. The author also explores how changes in political and social-economical structures might have influenced the development of the subject. To take some examples: in Chapter 4, the book explores the evolution of Chinese mathematics due to changing opinions in the Imperial Court of the Western missionaries who brought forth new ideas; in Chapter 5, the book also examines the state of mathematical education and how trigonometrical topics are implemented in the curriculum.

For any mathematicians and historians looking into the subject of trigonometry, this book is an invaluable resource that is faithful to both the source materials and the original authors. It achieves this loyalty whilst remaining accessible to the general audience and the entire book is further enriched by wonderful bits of commentaries and analysis on the background of each topic.

FURTHER READING

[1] Glen Van Brummelen, The mathematics of the heavens and the Earth: The early history of trigonometry, Princeton University Press, 2009.

Zhuo Zhang

Zhuo Zhang has recently finished his master's degree in Mathematics from the University of Oxford. He is interested in the history of mathematics, and in particular, ancient Chinese mathematics. Born in China, he came to the UK to study for his undergraduate degree. Chief amongst his pursuits now as a graduate student is cooking, as evidenced in his picture.

Obituaries of Members

Derek Goldrei: 1948–2022

Derek Goldrei, who was elected a member of the London Mathematical Society on 20 May 1983, died on 2 July 2022, aged 74.

Alex Wilkie writes: Derek was born on 2 July 1948 in London. His father Laurence owned and managed Ch. Goldrei, Foucard and Son, the bakery ingredient business established by his grandfather Charles, an immigrant from Russia. Derek's mother Irena arrived in the UK aged 18 from Vienna in 1938 after the Anschluss, at the invitation of his paternal grandmother, and she and Laurence married in 1942.

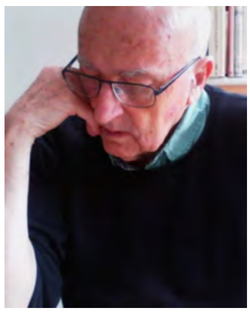
Derek attended St. Paul's School and Magdalen College, Oxford where in 1969 he was one of the first graduates in the newly created joint degree in Mathematics and Philosophy. He stayed in Oxford to study for a doctorate in Mathematical Logic under the supervision of Robin Gandy but actually spent much of this time working with Angus Macintyre and Harold Simmons at the University of Aberdeen on the model theory of Peano Arithmetic and its subtheories. We had a common research interest in this topic and I first met him in 1971 at the annual European Meeting of the Association for Symbolic Logic. We immediately struck up a friendship that lasted a lifetime.

Derek soon put pure research to one side and developed a preference for the various forms of teaching mathematics: writing courses and books as well as face-to-face tuition. So when he was offered a position at the newly created Open University he was ready and eager to take up the challenge. (This enthusiasm for his chosen career was also, I understand, conveyed to his father who, realising that his son would never want to take over the family business, decided to retire and sold it to Northern Foods.) One of his many achievements at the OU was to create and manage a third level course in Mathematical Logic which ran for over 25 years and which consistently attracted large numbers of students. I was hired to help write the course (together with Alan Slomson) and it was an inspiring experience, as well as a joy, to work with Derek. I particularly remember a rather ridiculous (I thought) afternoon at his house where we designed a cardboard cutout Turing machine kit! But it seemed to work and every student duly received one through the post alongside their course unit booklets and cassette tapes.

He worked for the OU all his life, becoming a Staff Tutor in 1978 and a Senior Lecturer in 1989. He wrote two influential textbooks: *Classic Set Theory* (1996) and *Propositional and Predicate Calculus: a Model of Argument* (2005), which will be, I am sure, on the reading lists of university logic courses for many generations to come. They have a unique style, rigorous but friendly, born of his many years of experience, in addition to his Open University work, of undergraduate teaching : he was a part-time lecturer at several Oxford colleges including Somerville (1978–2003) and, since 2003, Mansfield where he was giving revision tutorials just two months before his sudden and unexpected death on his 74th birthday.

I am grateful to Derek's sister Diane Goldrei and his wife Lindsey Court for providing some of the personal details noted above.

Stewart A. Robertson: 1933–2022



Professor Stewart A. Robertson, who was elected a member of the London Mathematical Society on 15 June 1961, died on 8 July 2022, aged 89. Professor Robertson was LMS Publications Secretary 1976-82.

David Chillingworth writes: A proud Scot, Stewart was born and raised in Broughty Ferry near Dundee. At school he excelled in most things, rising to Head Boy and Captain of the football team, and thought he could follow his brother as a joiner, or perhaps become a sea captain, although sound advice from school nudged him in a more academic direction. At age 17 he went to St Andrews University and was quickly promoted a year, later winning two prestigious undergraduate prizes. He stayed on to do a PhD with Edward Copson, and in his final year migrated with Copson to Leeds. In 1957 as a Leverhulme Research Fellow he moved to Liverpool where he stayed until 1970 as Lecturer and Senior Lecturer, with an intermediate year as

Visiting Professor at the University of California, Berkeley as well as a term at the University of Washington, Seattle. At Berkeley he was offered a tenure-track position, but chose to return home for family reasons.

For the rest of his academic career Stewart was Professor of Pure Mathematics at the University of Southampton. He led the way in a successful campaign to establish a separate Faculty of Mathematics (no longer extant), holding the post of Dean from 1978 to 1980, and was a Deputy Vice Chancellor 1984–1988, then and subsequently chairing many influential University reviews and committees including the public-facing Art Gallery and Concert Hall Committee. Ever the football enthusiast, he was also a robust member of a staff 5-a-side football team.

Stewart was involved in the foundation of the European Mathematical Society, co-chairing its Publications Committee in the earliest days of the EMS in the 1990s. On retirement in 1997 he was invited to serve on the Board of Southern Arts, and became a Governor at Portsmouth University as well as Chairman of the Aspex Gallery in Portsmouth. Although moving in elevated circles, however, he nevertheless took pleasure in gently and discreetly mocking those in authority. Always a witty and amusing raconteur, he particularly enjoyed reciting with a straight face the execrable poetry of William McGonagall.

Stewart's mathematical focus was mainly on differential and combinatorial geometry, and in 1982 he was awarded a prize by the Musée de la Villette, Paris for a beautifully-made collection of polyhedra representing all the symmetry types of a combinatorial cube, regrettably since recycled. He published around seventy articles in topology and geometry as well as a monograph *Polytopes and Symmetry* in the LMS Lecture Notes series. He supervised 20 successful PhD students of eight different nationalities, many simultaneously, putting to good purpose his remarkable skill in formulating problems of mathematical depth in relatively accessible topics.

Stewart was a man of wide culture in literature, philosophy, music, visual arts, theatre and sport as well as fine wine and (of course) whisky. He was an expert self-taught furniture restorer and maker, as well as a magnificent cook with emphasis on Mediterranean cuisine.

Whether at home or away on a coastal academic visit, he would often be down at the local fish market at dawn to get the best catch of the day.

In Liverpool he married Janet and they raised their two sons Alexander and Thomas in Southampton. He and Janet separated in 1987, and in 1991 Stewart married Debbie and they moved to her home town Tonbridge in 2003. Stewart was a kind and caring man, of sharp intellect, sadly blunted by the onset of Alzheimer's in his last years. Described by a friend as happy and rebellious right up to the end, when asked how he would cope with all the attention on his impending 80th birthday he said 'I've got it all worked out. I'm hiring an actor.'

Thanks to David Shairp and Debbie and Alex Robertson for important information on Stewart's life, and to several Southampton colleagues for valuable comments.

David Tipple: 1942–2022

Dr David Tipple, who was elected a member of the London Mathematical Society on 19 December 1968, died on 5 June 2022, aged 79.

Maria Meehan writes: David was born on 7 December 1942 in Manchester city, the only child of John and Mary Tipple. His father was a highly skilled craftsman who worked as tool-room turner. David was always proud of his working-class roots.

For his secondary education he attended Xaverian College, a grammar school in Manchester. A keen musician, with a special interest in classical music, he played the oboe in Manchester Youth Orchestra. He went on to study mathematics at the University of Manchester, obtaining a BSc and an MSc.

On completion of his master's degree, he secured a position as lecturer in University College Dublin (UCD), Ireland and moved there in 1967. As a young lecturer, he worked on his PhD which he was awarded in 1972 from the University of Manchester, with a thesis entitled *On the Metastable Homotopy Groups of Torsion Spheres* under the supervision of Professor Michael Barratt.

David joined UCD at a time of expansion and growth. The university moved from the city centre to the Belfield campus which was still a sprawling woodland. David and a close-knit group of colleagues kept logs of the birds they spotted on their lunchtime walks. This catalogue is now archived in UCD library.

In 1975 David met Fíona, a librarian newly arrived to UCD. They married in 1976 and enjoyed a further 46 years together. They both loved the Irish countryside and regularly holidayed there. His fondness for Ireland was apparent when he became an Irish citizen in 2005.

David was an active member of the Irish Mathematical Society, assuming the role of Treasurer from 1990–1993, and he played a significant role in redrafting the constitution of the society in 1992. He was an ardent supporter of student societies and enjoyed many years as Senior Treasurer of the university drama society, *DramSoc*.

An achievement that David was particularly proud of, was the role he played in introducing the Honours BA in Mathematical Studies to UCD. David felt a 3-year honours offering was required and was instrumental in the design and delivery of the new programme.

David prepared meticulously for lectures, and his courses were highly organised. Students held him in high regard, and he generously gave of his time to any student who appeared at his office door looking for assistance. In fact, he very much enjoyed these interactions.

I met David when I arrived as an assistant lecturer to UCD in 1997. He mentored me in all matters relating to my courses, patiently provided computer support, and of course corrected the grammar on my examination papers!

After his retirement in 2006, David pursued his interests with his usual vigour. An avid reader, with an interest in the history of typography and printing, he volunteered at Project Gutenberg, rising to the level of post-processor. He was most proud of a collaboration with his friend Prof Kevin Cathcart. They worked on *The Book of the Twelve Prophets* vol. II., by G. Adam, which had more than 1,500 footnotes, many in Greek and Hebrew; David ensured the fonts were faithfully reproduced.

David died after a short illness. Thank you to his wife Fíona, his friend Kevin Cathcart, his colleague Michael Mackey, and several of his other mathematics colleagues, for sharing their stories of David with me.

John McKay: 1939–2022

Professor John McKay, who was elected a member of the London Mathematical Society on 16 November 1967, died on 19 April 2022, aged 82.

Leonard Soicher writes: John McKay was an exceptional mathematician. His broad mathematical interests and knowledge, together with his curiosity, intuition, and insights, led to his discovery of important new connections between different areas of mathematics and the opening up of whole new areas of research. He was also a pioneer of the use of computers in pure mathematical research.

John McKay was born in Otford, Kent on 18 November 1939. He was a student at Dulwich College, and went on to study Mathematics at the University of Manchester where, after graduating, he remained as a Nuffield Research Fellow until 1964. During this period he worked with the Manchester Atlas computer, one of the most powerful computers in the world at that time.

In 1964 John moved to the University of Edinburgh to work on a PhD (in Computer Science) on algebra and computing with Sidney Michaelson and Douglas Munn. While at Edinburgh, he met and married Wendy, a fellow student. John left Edinburgh in 1967 to become a Research Fellow at the Atlas Computer Laboratory in Chilton, to work with another Atlas computer (one of three world-wide). Chilton was in commuting distance to Oxford, and John attended seminars in Mathematics at the University of Oxford. There he collaborated with Graham Higman, using his computer skills in the first constructions of the third sporadic simple group of Janko and the sporadic simple group of Held.

John McKay was excellent at connecting researchers and their ideas. In 1967–1968 John Leech was a Research Fellow at the Atlas Computer Laboratory, and was aware that his amazing 24-dimensional lattice would have a very large group of automorphisms. Leech was trying, with no luck, to get researchers in group theory interested in this automorphism group. However, John McKay was able to get John Conway interested, leading to Conway's discovery of his three sporadic simple groups.

In 1969 John McKay went to America as a Visiting Assistant Professor in Mathematics at the California

Institute of Technology (CalTech) where he completed and submitted his Edinburgh PhD thesis, primarily concerned with the computer calculation of the (ordinary) character tables of finite groups. John was formally awarded his PhD in 1971.

Around this time, the McKay Conjecture came into being, which concerns the number of irreducible complex characters of a finite group G that have degree not divisible by a given prime p. The Conjecture, which is still open, and its subsequent generalizations have driven major new developments in representation theory over the last 50 years.

In 1971 John took up a position in Computer Science at McGill University in Montreal, Canada, but he did not get on well there, and in 1974 moved up the road to become an Associate Professor in the Department of Computer Science at Concordia University.

In 1978, while working on computational Galois theory, John noticed that the co-efficient of q in the q-expansion of Felix Klein's j-invariant is $196884 = 1+196883$, and the smallest degrees of the irreducible complex representations of the (then conjectured) Monster group were 1 (for the trivial representation) and 196883. John thought this possible connection required investigation, which led in due course to the Conway-Norton monstrous moonshine conjectures, the proofs of which earned Richard Borcherds a Fields Medal in 1998.

In 1979 John made another remarkable discovery, now called the McKay Correspondence. This associates to a complex representation R of a finite group G a certain directed multigraph (now the *McKay graph*), and John observed that the graphs for the finite subgroups of $\mathrm{SU}(2,\mathbb{C})$ with respect to their natural 2-dimensional representations are precisely the extended A, D, and E Coxeter-Dynkin diagrams. This fact and its generalizations have since greatly influenced diverse areas of mathematics.

In 1979 John became a full Professor of Computer Science and in 1990 he took up a joint Computer Science and Mathematics position at Concordia. He was elected a Fellow of the Royal Society of Canada in 2000, and in 2003 won the CRM-Fields Prize for Mathematics, the highest honour for a mathematician in Canada. In 2007 a conference was jointly organised by Concordia University and the Centre de Recherches Mathématiques in Montreal to honour John McKay and celebrate his influence on mathematics.

In February 2017 John broke both his hips, and after four months in hospital was moved to a nursing home. There, with good humour, he kept up with the world and in touch with his colleagues, still having mathematical ideas. John died peacefully on 19 April 2022.

John and Wendy had divorced in 1985, and John married his second wife Trinh in 1988. John is survived by Trinh, his sister Elizabeth, his son Sacha and daughter Tanya from his first marriage, a nephew, and three grandchildren.

I was very fortunate to have John McKay as my supervisor for both my master's degree and postdoc. I thank Trinh Vo-McKay, John Harnad, Yang-Hui He, Hershy Kisilevsky, and Wendy McKay for interesting and useful information to help me write this obituary.

Death Notices

We regret to announce the following deaths:

- Derrick S.F. Crothers, formerly of Queen's University Belfast, who was elected a member of the London Mathematical Society on 15 June 1979, died on 15 January 2021, aged 78.
- Garth Dales, formerly of Lancaster University, who was elected a member of the London Mathematical Society 20 November 1969, died on 8 October 2022, aged 78.
- Geoffrey S. Joyce, of King's College London, who was elected a member of the London Mathematical Society on 19 November 1993, died on 18 June 2021, aged 80.
- David Monk, formerly of the University of Edinburgh, who was elected a member of the London Mathematical Society on 19 December 1957, died on 3 October 2022, aged 90.

Biographical Memoirs and LMS Obituaries

Memoirs for the following people have been published in *Biographical Memoirs of Fellows of the Royal Society:*

- Sir Erik Christopher Zeeman (1925–2016); tinyurl.com/dmbas73f
- Ludwig Dmitrievich Faddeev (1934–2017); tinyurl.com/2krpkurh

Obituaries (both recent and historical) published in the LMS *Bulletin* are free to read and can be accessed at tinyurl.com/bduxhkhe.

Mary Cartwright Lecture 2022

Location: ICMS, Edinburgh and online
Date: 10 November 2022
Website: lms.ac.uk/events/mary-cartwright-2022

The next LMS Society Meeting and Mary Cartwright Lecture will be held at the ICMS in Edinburgh on 10 November (2–6:30pm). The speakers will be Dan Margalit (Georgia Institute of Technology; *Reconstruction problems in mathematics: from Euclid to Ivanov*) and Tara Brendle (University of Glasgow; *Ivanov's metaconjecture: encoding symmetries of surfaces*). The event will be held in person and streamed online via Zoom. Sign up to attend at the link above.

Annual General Meeting 2022

Location: Goodenough College, London
Date: 18 November 2022, 3–6pm
Website: tinyurl.com/lms-agm2022

The meeting will begin with Society business, which will be followed by two lectures. The first lecturer, Josef Málek (Charles University, Prague), will speak on *Beyond the incompressible Navier-Stokes equations: mathematical foundations of models of non-Newtonian fluids*. The second lecturer, Endre Suli (Oxford), will give the annual Naylor Lecture on *Hilbert's 19th problem and discrete De Giorgi–Nash–Moser theory: analysis and applications*. The meeting will be followed by a Society Dinner.

Midlands Regional Meeting 2023

Location: Warwick
Date: 27–30 March 2023
Website: tinyurl.com/mid2023

The meeting forms part of the Midlands Regional Workshop on *Ergodic Theory of Group Extensions*, 28–30 March 2023. Speakers will be Viveka Erlandsson (Bristol University), Michael Magee (Durham University) and Mark Pollicott (Warwick University). The lectures are aimed at a general mathematical audience and all interested, whether LMS members or not, may attend. The meeting will be followed by a Society Dinner.

BCS/FACS Evening Seminar 2022

Location: Online, via Zoom
Date: 17 November 2022, 6–7:30pm
Website: lms.ac.uk/events/bcsfacs2022

In association with the British Computer Society – Formal Aspects of Computing Science (BCS-FACS), the LMS hosts an annual evening seminar on aspects of the computer science/mathematics interface. These events are free to anyone who wishes to attend, and have attracted high-quality speakers. The next LMS/BCS-FACS Evening Seminar will be held online, via Zoom, on Thursday 17 November. The speaker will be Professor Sam Staton (Oxford).

South West & South Wales Regional Meeting & Workshop

Location: Southampton
Date: 17 January 2023
Website: tinyurl.com/swsw2023

This meeting forms part of the South West & South Wales Workshop *Geometric Group Theory* in Southampton, 18–19 January 2023. The regional meeting's speakers will be Peter Kropholler (University of Southampton), Karen Vogtmann (University of Warwick) and Ian Leary (University of Southampton). It will be followed by a Society DInner. Funds are available for partial support for LMS members and research students; requests should be sent to Ian Leary (I.J.Leary@soton.ac.uk).

Society Meeting at the BMC 2023

Location: University of Bath
Date: 4 April 2023; 12 noon
Website: tinyurl.com/lms-bmc2023

The speaker at this meeting will be Tim Browning (Institute of Science and Technology Austria). The talk will be aimed at a general mathematical audience and the meeting is open to both LMS members and non-members. Members will be able to sign the Members' Book, which dates from 1865. Registration for the BMC is now open; the early bird rate (lasting until December 2022) is £75.

Society Meetings and Events

November 2022

10 Society Meeting and Mary Cartwright Lecture, ICMS, Edinburgh and online
17 LMS–BCS/FACS Evening Seminar, online
18 LMS Annual General Meeting and Naylor Lecture, London

January 2023

17 LMS South West and South Wales Meeting and Workshop, University of Southampton

March 2023

27-30 LMS Midlands Regional Meeting, Warwick

April 2023

4 LMS Meeting at the BMC 2023, Bath

Calendar of Events

This calendar lists Society meetings and other mathematical events. Further information may be obtained from the appropriate LMS Newsletter whose number is given in brackets. A fuller list is given on the Society's website (www.lms.ac.uk/content/calendar). Please send updates and corrections to calendar@lms.ac.uk.

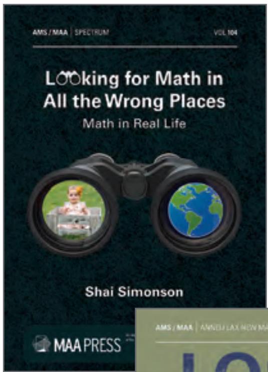
December 2022

16-17 Random Matrix Theory, Brunel University, London

April 2023

3-6 British Mathematical Colloquium, University of Bath (503)

AMS AMERICAN MATHEMATICAL SOCIETY



LOOKING FOR MATH IN ALL THE WRONG PLACES

Math in Real Life

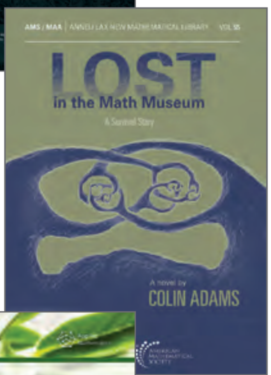
Shai Simonson, Stonehill College

Spectrum, Vol. 104

The soul of mathematics is the practice of skeptical inquiry: asking how and why things work, experimenting, exploring, and discovering. Estimation, analysis, computation, conjecture, and proof are the mathematical path to uncovering truth and we can use them in nearly every human pursuit. In this thoroughly charming and beguiling book, Shai Simonson applies mathematical tools in a variety of contexts that arise in everyday life to prove his claim that math is, literally, everywhere.

Sep 2022 209pp 9781470470128 Paperback £32.50

MAA Press



LOST IN THE MATH MUSEUM

A Survival Story

Colin Adams, Williams College

Anneli Lax New Mathematical Library, Vol. 55

From the twisted imagination of bestselling author Colin Adams comes this tale of sixteen-year-old Kallie trying to escape death at the hands of the exhibits in a mathematics museum. Kallie crosses paths with Carl Gauss, Bertrand Russell, Sophie Germain, G. H. Hardy, and John von Neumann, as she tries to save herself, her dad, and his colleague.

Oct 2022 209pp 9781470468583 Paperback £32.50

MAA Press

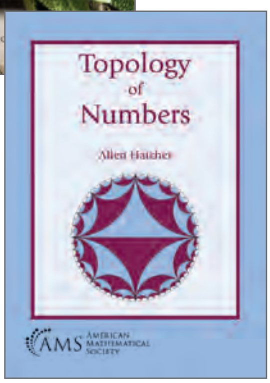
SAGE FOR UNDERGRADUATES

Second Edition, Compatible with Python 3

Gregory V. Bard

A gentle introduction to Sage for undergraduate students during Calculus II, Multivariate Calculus, Differential Equations, Linear Algebra, Math Modeling, or Operations Research. In 2019, Sage transitioned from Python 2 to Python 3, which changed the syntax in several significant ways, including for the print command. All the examples in this book have been rewritten to be compatible with Python 3.

Oct 2022 481pp 9781470461553 Paperback £54.95



TOPOLOGY OF NUMBERS

Allen Hatcher, Cornell University

An introduction to number theory at the undergraduate level, emphasizing geometric aspects of the subject. The geometric approach is exploited to explore in some depth the classical topic of quadratic forms with integer coefficients, a central topic of the book. Quadratic forms of this type in two variables have a very rich theory, developed mostly by Euler, Lagrange, Legendre, and Gauss during the period 1750–1800. In this book their approach is modernized by using the splendid visualization tool introduced by John Conway in the 1990s called the topograph of a quadratic form.

Nov 2022 349pp 9781470456115 Paperback £54.95

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