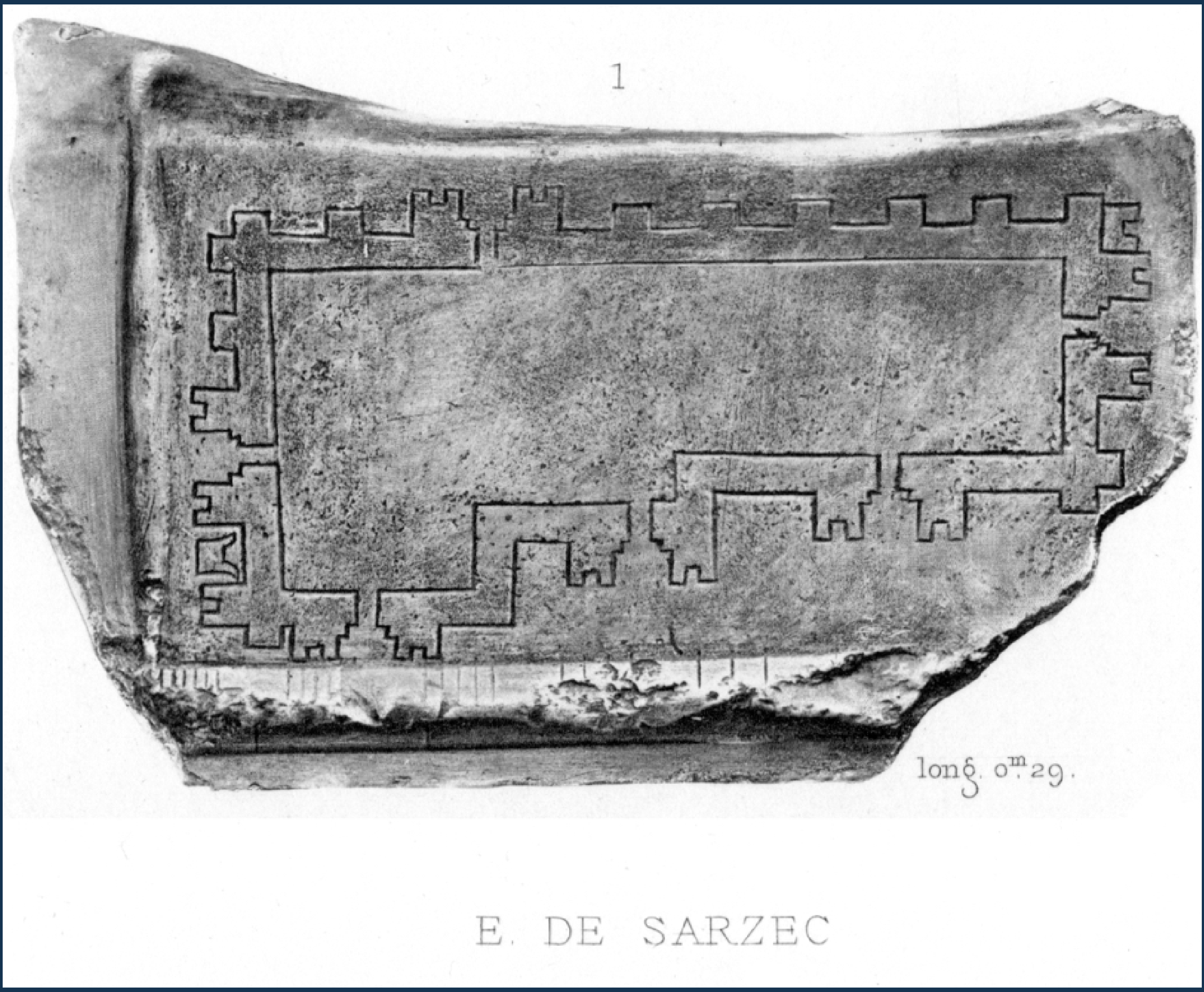
LONDON MATHEMATICAL SOCIETY
EST. 1865

NEWSLETTER

Issue: 508 - September 2023



PROJECTIVE GEOMETRY AND TENSOR SPACES

GUDEA'S MEASURING ROD

NOTES OF A NUMERICAL ANALYST

EDITORIAL OFFICE

London Mathematical Society
De Morgan House
57–58 Russell Square
London WC1B 4HS
newsletter@lms.ac.uk

Charity registration number: 252660

COVER IMAGE

Figure 1 ('Gudea temple plan') from the article 'Gudea's Measuring Rod', page 34.

Do you have an image of mathematical interest that may be included on the front cover of a future issue? Email images@lms.ac.uk for details.

ISSN: 2516-3841 (Print)
ISSN: 2516-385X (Online)
DOI: 10.1112/NLMS

NEWSLETTER WEBSITE

The Newsletter is freely available electronically at lms.ac.uk/publications/lms-newsletter.

MEMBERSHIP

Joining the LMS is a straightforward process. For membership details see lms.ac.uk/membership.

SUBMISSIONS

The Newsletter welcomes submissions of feature content, including mathematical articles, career related articles, and microtheses from members and non-members. Submission guidelines and LaTeX templates can be found at lms.ac.uk/publications/submit-to-the-lms-newsletter.

Feature content should be submitted to the editor-in-chief at newsletter.editor@lms.ac.uk.

News items should be sent to newsletter@lms.ac.uk.

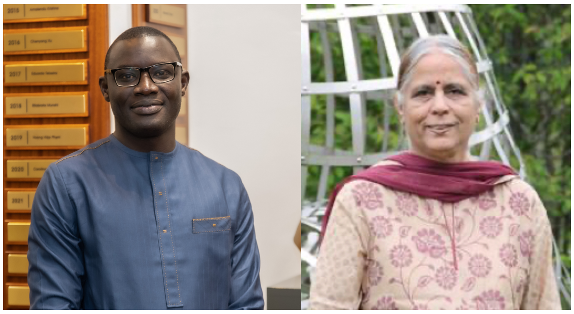
Notices of events should be prepared using the template at lms.ac.uk/publications/lms-newsletter and sent to calendar@lms.ac.uk.

For advertising rates and guidelines see lms.ac.uk/publications/advertise-in-the-lms-newsletter.

CONTENTS

LMS NEWS

LMS Honorary Members 2023



Professor Mouhamed Moustapha Fall and Professor Raman Parimala (Image source for RP: The Oberwolfach Photo Collection)

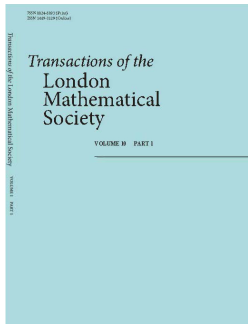
The LMS has elected Professor Mouhamed Moustapha Fall, President and Professor at AIMS Senegal, and Professor Raman Parimala, Arts and Sciences Distinguished Professor of Mathematics at Emory University, USA, to Honorary Membership of the Society in 2023.

Professor Fall has made outstanding contributions in the fields of partial differential equations and differential geometry, in particular through fundamental results concerning the existence and non-existence of solutions of linear and nonlinear partial differential equations inspired by deep questions in mathematical physics and differential geometry. He is also highly recognised for his support for the advancement of the mathematical sciences in Africa through his outstanding research and public engagement. In recognition of his many contributions, he was awarded the 2022 Ramanujan Prize and was also an invited speaker at the 2018 ICM in Rio de Janeiro.

Professor Parimala has made fundamental contributions to the theory of quadratic forms, Galois cohomology, and the theory of principal bundles for algebraic groups. Among her many achievements is her proof with Eva Bayer-Fluckiger of the triviality of principal bundles for classical algebraic groups over fields of cohomological dimension at most two, a deep and difficult result that had been conjectured by Jean-Pierre Serre. Professor Parimala's research has been distinguished throughout her career by her amalgamation of arithmetic, algebra, and geometry, whereby intricate constructions at the interface of the three domains lead to the resolution of difficult problems of both classical and modern nature.

Full citations will appear in the LMS *Bulletin*.

Transactions — a Journal to Consider



A couple of years ago we relaunched the *Transactions of the London Mathematical Society* (*Newsletter*, January 2021) with a stellar Editorial Board which now includes three Fields Medallists among a range of distinctive, creative, leading mathematicians (see below). The journal is now well established, having acquired its first Impact Factor this year as a testament to its success.

The *Transactions* was originally conceived as the LMS's Open Access journal, but in recent years the landscape and nature of publishing have been shifting around it — not only on the matter of Open Access, but also in the roles and identities of the LMS's other journals (*Newsletter*, July 2023).

In relation to Open Access, our aim clearly states that the *Transactions* should be free of charge both to read and to publish in, and we have a range of measures in place to achieve this. Also important is that the *Transactions* should have a clearly defined long-term role that will transcend and survive changes in the surrounding landscape. In this spirit, the scope of the *Transactions* emphasises "excellent exposition of research which explores the interconnectedness of pure mathematics or extends the boundaries of its applicability."

Authors will naturally consider many criteria when deciding where to publish. But if you have a piece of work — perhaps distinctive, perhaps a little different — which fits the description above, and which you think might appeal to one of our Board members, we would love to receive it. To submit your paper, visit lms.ac.uk/tlms.

The current Editorial Board includes Gang Bao (China), Amin Coja-Oghlan (Germany), Jian Ding (China), Charles Fefferman (USA), Alessio Figalli (Switzerland), Jelena Grbic (UK), Martin Hairer (UK), Heather Harrington (UK), John King (UK), Jessica Purcell (Australia), José Rodrigo (UK), Carola-Bibiane Schönlieb (UK), Sebastian van Strien (UK), Marcelo Viana (Brazil), Juncheng Wei (Canada), Sarah Zerbes (Switzerland) and Ping Zhang (China).

José Rodrigo
Managing Editor, *Transactions of the LMS*

LMS Prize Winners 2023

The 2023 LMS prize winners were announced at the Society Meeting on Friday 30 June 2023. The LMS extends its congratulations to this year's prize winners for their continued contributions to mathematics.

Professor Dame Frances Kirwan FRS of the University of Oxford is awarded the Pólya Prize for her many outstanding and influential results in geometry and for her career-long service to the mathematical community.

Professor Agata Smoktunowicz of the University of Edinburgh is awarded a Senior Whitehead Prize for her agenda-setting work in ring theory, which single-handedly reawakened interest in classical problems that had been moribund for many years. Other themes in her research have provided fundamental underpinning to noncommutative (projective) algebraic geometry.

Dr Eugénie Hunsicker of the Access Group is awarded the Senior Anne Bennett Prize for her outstanding work to improve equality and diversity in the mathematical community and for the depth of her mathematical achievements across an impressive range of areas, from L2 Hodge theory to data science.

Professor Jens Eggers of the University of Bristol is awarded a Naylor Prize and Lectureship for his profound contributions to the theoretical understanding of singularities of nonlinear partial differential equations, particularly his construction of explicit solutions of equations that yield deep insight into experiments.

Dr Jian Ding of Peking University and Dr Ewain Gwynne of the University of Chicago are awarded a Berwick Prize for their paper *Uniqueness of the Critical and Supercritical Liouville Quantum Gravity Metrics*, published in the *Proceedings of the LMS*. This paper completes the mathematical construction and characterisation of the physically natural random metrics in the plane defined via exponentials of the Gaussian Free Field in the delicate cases where it turns out that a dense set of singular points are at infinite distance from all others.

Dr David Bate of the University of Warwick is awarded a Whitehead Prize for his deep and fundamental contributions to the development of geometric measure theory in the metric setting, including the characterisations of rectifiability in terms of projections and in terms of tangent planes.

Dr Soheyla Feyzbakhsh of Imperial College London is awarded a Whitehead Prize for her spectacular applications of wall-crossing techniques to questions in classical and enumerative algebraic geometry.

Professor András Juhász of the University of Oxford is awarded a Whitehead Prize for his fundamental work in low-dimensional topology, in particular for applying Heegaard Floer homology to obtain a better understanding of knots, 3-manifolds, and 4-manifolds.

Professor Mahesh Kakde of the Indian Institute of Science in Bangalore is awarded a Whitehead Prize for his outstanding contributions to Iwasawa theory and the solution of famous and important conjectures in number theory related to zeta and L-values, and for enormous advances on Hilbert's 12th problem.

Dr Yankı Lekili of Imperial College London is awarded a Whitehead Prize for his illuminating and wide-reaching work in symplectic topology and homological mirror symmetry, and their connections to representation theory and arithmetic geometry.

Professor Marie-Therese Wolfram of the University of Warwick is awarded a Whitehead Prize for her groundbreaking contributions to applied partial differential equations, mathematical modelling in socio-economic applications and the life sciences, and numerical analysis of partial differential equations.

2023 LMS Prize Winners

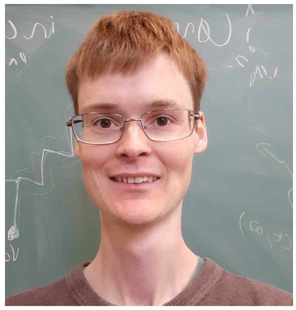

Frances Kirwan
Pólya Prize


Agata Smoktunowicz
Senior Whitehead Prize


Eugénie Hunsicker
Senior Anne Bennett Prize


Jens Eggers
Naylor Prize & Lectureship


Jian Ding
Berwick Prize


Ewain Gwynne
Berwick Prize


David Bate
Whitehead Prize


Soheyla Feyzbakhsh
Whitehead Prize

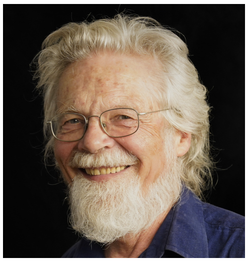

András Juhász
Whitehead Prize


Mahesh Kakde
Whitehead Prize


Yanki Lekili
Whitehead Prize


Marie-Therese Wolfram
Whitehead Prize

Hirst Prize and Lectureship Winner 2023



The 2023 Hirst Prize and Lectureship has been awarded to Professor Erhard Scholz, Emeritus Professor of the History of Mathematics at the University of Wuppertal. The Hirst Prize is awarded jointly with the LMS and the British Society for the History of Mathematics. Professor Scholz's distinguished academic career has spanned five decades. He has worked in many fields, always with high standards of scholarship and clarity, and is widely regarded as a leading historian of mathematics.

LMS–IMA Crighton Medal Winner 2023

The 2023 David Crighton Medal, awarded jointly by the LMS and the IMA, has been awarded to Alison Etheridge FRS OBE, of the University of Oxford.

The David Crighton Medal was established to pay tribute to the memory of Professor David George Crighton FRS for services both to mathematics and to the mathematical community.

Professor Etheridge is a leading researcher and a world expert on stochastic processes and their applications. She is an excellent lecturer, and her skills as a researcher and educator have attracted over 30 graduate students and postdocs over the years.

Professor Etheridge has played an active role in encouraging women in mathematics, and has provided excellent guidance and mentoring to female PhD students and junior colleagues, who now have successful careers in their own right. By sharing her experiences about her research and career path, she has shown how she balanced her career and family life, providing suggestions on how partners and employer of women in mathematics can be supportive of them.

Through all of the above, and much more, Professor Etheridge has been an effective role model and mentor for women in mathematics; her appointments in important roles during her career signal the high esteem in which she is held and the mathematical community's recognition of her dedication and abilities as a leader in her field.

Emmy Noether Fellows 2023

The LMS Committee for Women and Diversity in Mathematics is pleased to announce the winners of the second round of Emmy Noether Fellowship Grants for 2022/23:

- Dr Anthea Monod, Imperial College London
- Dr Anne-Sophie Kaloghiros, Brunel University
- Dr Hong Qi, Queen Mary University of London
- Dr Ana Peón-Nieto, University of Birmingham
- Dr Xiaocheng Shang, University of Birmingham
- Dr Sara Frecentese, University of Liverpool

The fellowships are designed to enhance the mathematical sciences research, broadly construed, of holders either re-establishing their research programme after returning from a major break associated with caring responsibilities or those requiring support to maintain their research programme while dealing with significant ongoing caring responsibilities.

The Emmy Noether Fellowship Grant is made possible by the continued generosity of the Liber Stiftung (Liber Foundation).

We expect to open the next round in early 2024. For further information on the Emmy Noether Fellowships, please visit https://bit.ly/emmy-noether-fellowships.

Forthcoming LMS Events

The following events will take place in forthcoming months:

LMS Northern Regional Meeting & Workshop: 4-6 September, University of York (bit.ly/nrm-23)

Black Heroes of Mathematics Conference 2023: 3-4 October, online (bit.ly/bhom-2023)

Joint Meeting with the IMA: 13 October, London (bit.ly/joint-meeting-2023)

Mary Cartwright Lecture and Society Meeting: 19 October, ICMS Edinburgh (bit.ly/mary-cartwright-2023)

Annual General Meeting & Presidential Address 2023: 17 November, London (bit.ly/3OdmOKJ)

A full listing of upcoming LMS events can be found on page 46.

Growing the LMS Membership

The LMS Council encourages all mathematicians and mathematics students based in the UK and overseas to join the LMS and support its charitable aims in advancing, disseminating and promoting mathematical knowledge. The LMS also very much supports applications from mathematicians who are working or have worked in other occupations.

If current members know of friends or colleagues who would like to join the Society, please encourage them to complete the online application form at lms.ac.uk/membership/online-application. As a current member, you can act as the proposer or seconder for their application.

LMS members benefit from membership of a vibrant national and international mathematics community, with opportunities to apply for research grants, attend Society events and influence national policy. In addition to receiving regular Newsletters and e-updates, LMS members enjoy exclusive access to the Verblunsky Members' Room, free online subscription to selected journals and complimentary use of the Society's Library at UCL, among other benefits (see lms.ac.uk/membership/member-benefits).

Contact membership@lms.ac.uk for advice on becoming an LMS member.

New Membership Rates: Lifetime and Retired

The Society's new Lifetime and Retired membership rates have been introduced to accommodate the evolving needs and circumstances of our diverse community.

Lifetime Membership

Many of our members have shown long-lasting commitment over the years. In recognition of their continued support, we now offer a Lifetime Membership option. In choosing this rate, members can enjoy access to all LMS events, facilities and other benefits without the need for annual renewals. With a one-time payment, lifetime members will be exempt from future price adjustments, which will lead to significant savings over the long term.

Retired Membership

We understand that retirement marks a significant transition in one's life, and we want to ensure that our retired members can continue to enjoy the benefits of our community during this new phase. To this end, our Retired Membership is available to all current members who have retired from their professional careers.

If you would like to upgrade to these new membership rates, please contact our membership team: membership@lms.ac.uk.

Renew Your LMS Membership in 2023/24

The standing and usefulness of the Society depends upon the support of a strong membership to provide the resources, expertise and participation in the running of the Society across its many activities in publishing, grant-giving, conferences, public policy, influencing government, and mathematics education in schools.

Renewal reminders about annual membership fees, including information on how to make payment for additional subscriptions, for the period November 2023–October 2024 will be sent either by email or by post in late September/early October 2023. Annual subscriptions become due on 1 November 2023 and payment should be received by 1 December 2023. Please note that payments received after this date may result in a delay in journal subscriptions being renewed.

Tiered Ordinary Membership Subscription Rates

Members who pay the Ordinary membership rate can choose which membership rate they wish to pay based on whether their annual professional income falls within the following ranges:

- Above £70,000 per annum: Ordinary (high) member rate.
- Between £40,000-£70,000 per annum: Ordinary (middle) member rate.
- Up to £40,000 per annum: Ordinary (low) member rate.

The Society will not collect any data on members' actual professional income nor require proof of earnings. Instead, Ordinary members are asked to advise either via their online member record or the subscription form which tier of Ordinary membership subscription they will be paying. For members who pay by direct debit, we encourage you to update this information by 14 October 2023.

All LMS Membership Fee Rates

The annual membership fee rates for the London Mathematical Society in 2023-24 can be seen below.

Renewal and Payment

Online:
Members can log on to their LMS user account (lms.ac.uk/user) to make changes to their contact details and journal subscriptions, and to make payment either by card via WorldPay or by setting up a direct debit via GoCardless, under the "My LMS Membership" tab.

By Subscription Form:
Members can also renew their subscription by completing the subscription form and including a cheque either in GBP or USD. We regret that we do not accept payment by cheques in Euros.

Queries

Please do email any queries to the LMS membership team (membership@lms.ac.uk) or call us; 020 7927 0808 (Monday – Tuesday) or 020 7291 9973 (Wednesday – Friday).

LMS Membership Subscription Rates 2021-22		
Ordinary Member (high) rate	£130.00	US$260.00
Ordinary Member (middle) rate	£106.00	US$212.00
Ordinary Member (low) rate	£86.25	US$172.50
Reciprocity rate for members based outside the UK and a member of one of the LMS' Reciprocity partners	£53.00	US$106.00
*Associate (postdoc) rate for early career members whose PhD completion was more than 3 years ago and who are on a non-permanent contract	£53.00	US$106.00
*Retired rate for members who have retired and are building towards Senior (free) membership	£53.00	US$106.00
Associate rate for PhD student members and members whose PhD was completed in the last three years	£26.50	US$53.00
*Concessionary rate for members working part-time, unemployed or otherwise in hardship	£26.50	US$53.00
Associate (Undergraduate) rate for undergraduate student members	£13.25	US$26.50
Senior rate for members who have paid fees for at least 35 years	£0.00	US$0.00

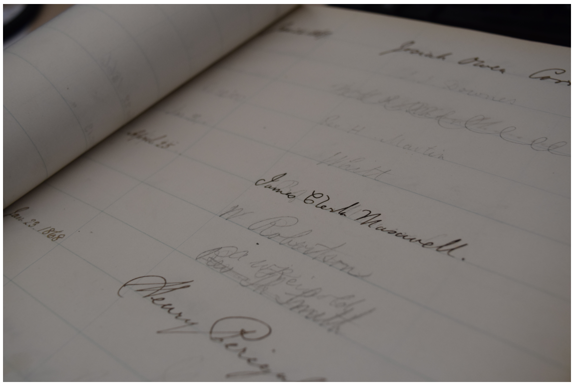
*These rates are by request and subject to agreement by the Treasurer.

For the Lifetime Membership rate for members who wish pay a one-off lump to cover the remainder of their 35 years' of contributions, please contact the Membership Team for a quote.

FOCUS ON LMS MEMBERSHIP

Maximising your LMS Membership

Signing the LMS Members' Book



A unique benefit of LMS membership is the opportunity for members to sign the LMS Members' Book. After election to membership, and once they have paid their subscription fees, members can sign their name into the history of the LMS.

In a leather-bound volume that dates from the Society's founding in 1865, the LMS Members' Book provides a physical link between current members and the first members of the Society whose signatures, such as Augustus De Morgan, appear in the early pages. Leafing through, and one finds the signatures of other well-known mathematicians such as M.L. Cartwright, G.H. Hardy, Felix Klein, Henri Poincaré, are inscribed within the first half of the Members' Book.

While it's been nearly 160 years since the first autographs, just over half the pages have been filled, which leaves plenty of room for any members, who have not yet signed, to add their name. The LMS Members' Book appears at every in-person Society Meeting, as far as is practicable, and it has travelled across the globe so members based outside the UK are included.

Your next opportunities to sign the LMS Members' Book in 2023-24 will be:

- 4 September 2023 — Northern Regional Meeting, King's Manor in York
- 13 October 2023 — LMS-IMA Joint Meeting, De Morgan House in London
- 19 October 2023 — LMS Mary Cartwright Lecture & Society Meeting, ICMS in Edinburgh
- 17 November 2023 — LMS AGM & Presidential Address, Mary Ward House in London
- 17 January 2024 — LMS South West & South Wales Regional Meeting, Bath University

At each of these meetings, we will invite those members who can attend in-person, and who have not yet done so, to come forward when asked by the President and sign the LMS Members' Book.

Voting in the LMS Elections

The LMS Council encourages all members to vote in the annual elections to Council and Nominating Committee, which this year will take place in October and November. The slate of candidates can be found on the LMS website at lms.ac.uk/about/council/lms-elections and all members are invited to contribute to an online discussion forum, which is at discussions.lms.ac.uk/lmselections.

Instructions on how to vote will be sent to members by e-mail or post by Civica Election Services on 13 October 2023, the day voting opens. Members are encouraged to check that their contact details are up to date at lms.ac.uk/user.

The results of the Council and Nominating Committee elections will be announced at the Annual General Meeting on 17 November at 3.00pm.

The Society promotes a fair and transparent election that benefits the mathematical community and portrays a positive public image of its members. Competitive elections allow the Society to form a strong governing body responsible for the general control and management of its administration, strategies and financial operations.

OTHER NEWS

Visual Mathematical Dictionary

The Visual Mathematical Dictionary is a website that has collated and translated dozens of mathematical terms into more than 30 languages, helping to make the learning process accessible to across different linguistic backgrounds.

It has been developed by Prof. Antonella Perucca and Dr. Olha Nesterenko in the Department of Mathematics at the University of Luxembourg. The dictionary is completely free and does not contain any logos or commercial affiliation, nor require any registration or a third-party app to access. As the creators state, “we believe in the power of knowledge-sharing and want to ensure that our resources are readily available to anyone”.

Far from being complete, the developers intend to update the website, adding new chapters, expanding the content further and updating the materials where needed.

The Visual Mathematical Dictionary is available at math.uni.lu/dictionary.

MATHEMATICS POLICY DIGEST

Ofsted Report into Maths Teaching

A new subject report into maths teaching, *Coordinating Mathematical Success,* has been published by Ofsted in July.

The full report identifies a range of points, including:

- Schools can benefit from adopting a consistent approach to designing and implementing the curriculum, with an emphasis on content and ‘small steps’ sequencing
- Pupils need to learn strategies to solve different types of problems, and teaching should be planned to allow them to develop a range of strategies over time so that they are able to select an effective and appropriate strategy
- Teachers, including non-specialists, should receive the necessary professional development, including subject knowledge and subject-specific pedagogical knowledge, to teach maths effectively
- Schools and teachers should focus on securing learning, only moving on to the next conceptual step when pupils are ready, whilst also not limiting access to further mathematical content based upon external assessment criteria.

Read the report in full at bit.ly/ofsted-maths-report.

JMC Report on Maths Education and Digital Technology

The Joint Mathematical Council (JMC) recently published its 2023 report on Digital Technologies and Mathematics Education, a follow-up to the original report published in September 2011. The 2023 report reflects on the changes in technology and education in the preceding years since the first report and considers the significant effects of the global pandemic on these aspects.

The report notes that the landscape of digital technology is highly complex, acknowledging that significant systemic change is difficult to achieve. It states that despite the promises of digital technology to enhance mathematics education, and the ongoing transformation of all aspects of modern society by technology, little has changed in the intervening years since the publication of the 2011 Report. The authors state that progress against its recommendations has been slow at best.

Read the report at bit.ly/jmc-report-2023.

Digest prepared by Kieran O’Connor
Events Co-ordinator

Note: items included in the Mathematics Policy Digest are not necessarily endorsed by the Editorial Board or the LMS.

Open Call for Caring Grant Applications

The LMS Committee for Women and Diversity in Mathematics offers Caring Supplementary Grants to enable parents and carers to attend conferences, research schools, meetings or visits by making a contribution towards caring costs. Any mathematician in the UK is eligible to apply and the maximum award is £200.

Applications for Caring Supplementary Grants in the academic year 2023/24 are open year-round. There is no deadline, but decisions are made on a quarterly basis. It is expected that the first round of decisions will be made in September 2023. If an application is not successful in one round, it may be considered in the next.

Many LMS research grant schemes include provision for caring costs. Applicants who have received a research grant with caring costs included will not be eligible to apply for a Caring Supplementary Grant for the same activity.

Applications should be made using an application form and sent to the grants administrator, Kieran O'Connor: womenanddiversity@lms.ac.uk.

Read more and apply at bit.ly/caring-23-24.

LMS Research and Travel Grants

Deadline: 15 September 2023

Applications are invited for the following grant schemes to be considered in October 2023. Applicants should be mathematicians based in the UK, the Isle of Man or the Channel Islands. For grants to support conferences/workshops, the event must be held in the UK, the Isle of Man or the Channel Islands.

See details of these schemes, including how to apply, at lms.ac.uk/grants/research-grants.

Conferences (Scheme 1)

Grants of up to £5,500 are available to provide partial support for conferences. This includes travel, accommodation and subsistence expenses for principal speakers, UK-based research students, participants from Scheme 5 countries and Caring Costs for attendees who have dependents.

Visits to the UK (Scheme 2)

Grants of up to £1,500 are available to provide partial support for a visitor who will give lectures in at least three separate institutions. Awards are made to the host towards the travel, accommodation and subsistence costs of the visitor. Potential applicants should note that it is expected the host institutions will contribute to the costs of the visitor. In addition, the Society allows a further amount (of up to £200) to cover Caring Costs for those who have dependents.

Joint Research Groups in the UK (Scheme 3) — Application deadline 30 September 2023

Grants of up to £1,500 are available to support joint research meetings held by mathematicians who have a common research interest and who wish to engage in collaborative activities, working in at least three different locations (of which at least two must be in the UK).

Research in Pairs (Scheme 4)

For those mathematicians inviting a collaborator, grants of up to £1,200 are available to support a visit for collaborative research either by the grant holder to another institution abroad, or by a named mathematician from abroad to the home base of the grant holder. For those mathematicians collaborating with another UK-based mathematician, grants of up to £600 are available to support a visit for collaborative research either by the grant holder to another institution or by a named mathematician to the home base of the grant holder. In addition, the Society allows a further amount (of up to £200) to cover Caring Costs for those who have dependents.

Research Reboot (Scheme 4)

Grants of up to £500 for accommodation, subsistence and travel plus an additional £500 for caring costs are available to assist UK mathematicians who may have found themselves with very little time for research due to illness, caring responsibilities, increased teaching or administrative loads, or other factors. This scheme offers funding so that they can leave their usual environment to focus entirely on research for a period from two days to a week. For applications submitted by the next deadline (15 September 2022), the Reboot Retreats should take place between 01 November 2022 and 30 January 2023.

Collaborations with Developing Countries (Scheme 5)

For those mathematicians inviting a collaborator to the UK, grants of up to £3,000 are available to support a visit for collaborative research, by a named mathematician from a country in which mathematics could be considered to be in a disadvantaged position, to the home base of the grant holder. For those mathematicians going to their collaborator's institution, grants of up to £2,000 are available to support a visit for collaborative research by the grant holder to a country in which mathematics could be considered to be in a disadvantaged position. Applicants will be expected to explain in their application why the proposed country fits the circumstances considered eligible for Scheme 5 funding. In addition, the Society allows a further amount (of up to £200) to cover Caring Costs for those who have dependents. Contact the Grants team if you are unsure whether the proposed country is eligible or check the IMU's Commission for Developing Countries definition of developing countries (tinyurl.com/y9dw364o).

Research Workshop Grants (Scheme 6)

Grants of up to £10,000 are available to provide support for Research Workshops. Research Workshops should be an opportunity for a small group of active researchers to work together for a concentrated period on a specialised topic. Applications for Research Workshop Grants can be made at any time but should normally be submitted at least six months before the proposed workshop.

African Mathematics Millennium Science Initiative (AMMSI)

Grants of up to £2,000 are available to support the attendance of postgraduate students at conferences in Africa organised or supported by AMMSI. Please contact grants@lms.ac.uk for more information.

The next closing date for early career research grant applications (Schemes 8,9 and ECR Travel Grants) is 15 October 2023. Applications are invited for the following grants to be considered by the Early Career Research Committee at its November 2023 meeting:

Postgraduate Research Conferences (Scheme 8)

Grants of up to £2,500 are available to provide partial support for conferences, which are organised by and are for postgraduate research students. The grant award will be used to cover the costs of participants. In addition, the Society allows the use of the grant to cover to cover Caring Costs for those who have dependents.

Celebrating New Appointments (Scheme 9)

Grants of up to £400-£500 are available to provide partial support for meetings to celebrate the new appointment of a lecturer at a university. Potential applicants should note that it is expected that the grant holder will be one of the speakers at the conference. In addition, the Society allows the use of the grant to cover to cover Caring Costs for those who have dependents.

ECR Travel Grants

Grants of up £500 are available to provide partial travel and/or accommodation support for UK-based Early Career Researchers to attend conferences or undertake research visits either in the UK or overseas.

LMS Research Schools and Research Schools on Knowledge Exchange

Deadline for Research Schools in 2024: 15 September 2023

Deadline for Research Schools in 2025: 22 February 2024

Grants of up to £15,000 are available for LMS Research Schools in 2024 and 2025, one of which in each year will be focused on Knowledge Exchange. The Research Schools provide training for research students in all contemporary areas of mathematics. The Knowledge Exchange Research Schools will primarily focus on Knowledge Exchange and can be in any area of mathematics.

The Research Schools take place in the UK and support participation of research students from both the UK and abroad. The lecturers are expected to be international leaders in their field. The LMS Research Schools are often partially funded by the Heilbronn Institute for Mathematical Research (http://heilbronn.ac.uk). Information about the submission of proposals can be found at tinyurl.com/ychr4lwm along with a list of previously supported Research Schools. Applicants are strongly encouraged to discuss their ideas for Research Schools with the Chair of the Early Career Research Committee, Professor Chris Parker (research.schools@lms.ac.uk) before submitting proposals. Proposals should be submitted to Lucy Covington (research.schools@lms.ac.uk).

To extend the international reach of the Research Schools, prospective organisers may also wish to consider applying to the Clay Mathematics Institute (CMI) for additional funding under the CMI's Enhancement and Partnership Program. Further information about this program can be found at bit.ly/3YbwWZf. Prospective organisers are advised to discuss applications to this program as early as possible by contacting the CMI President, Martin Bridson (president@claymath.org). There is no need to wait for a decision from the LMS on your Research School application before contacting the CMI about funding through this programme.

Diversity in Maths Events 2023/24: Call for Expressions of Interest

Deadline: 1 October 2023 (11:59pm)

The LMS Committee for Women and Diversity in Mathematics invites expressions of interest from individuals or groups to organise and host events focused on diversity in mathematics in the academic year 2023/24. These events may fall into one of the following categories:

Women and Non-Binary People in Mathematics Day (up to £1,000 per grant)

A Women and Non-Binary People in Mathematics Day should aim to help early career women (including trans women) and non-binary mathematicians who are considering the next stages in their careers. It typically includes mathematical talks combined with panel discussions and networking opportunities.

Read more and apply at bit.ly/wnb-day-2023.

Diversity in Mathematics Day (up to £1,000 per grant)

The Diversity in Mathematics Day should focus on some aspect of diversity in the mathematical sciences beyond gender. The event should feature the work of people in the mathematical sciences, whether in industry or academia, who come from a particular diversity group, and also offer opportunities for mathematicians from that diversity group for mentoring and networking.

Applicants typically come from a university or from a school, although any qualified person may apply. If the event is to be held in a school, it should encourage students from diverse backgrounds to become more involved in mathematics.

Read more and apply at https://bit.ly/diversity-day-23.

Girls in Mathematics Day (up to £500 per grant)

A Girls in Mathematics Day should be aimed at school students, up to and including A-level or equivalent, with mathematics as a main focus. There is flexibility regarding the format of the day but the event should include varied activities aimed at inspiring and encouraging girls to explore mathematics.

Read more and apply at bit.ly/gim-2023.

Send queries and completed applications to womenanddiversity@lms.ac.uk.

Cecil King Travel Scholarship: Call for Applications

Deadline: 15 November 2023

Applications for the Cecil King Travel Scholarships 2024 are now open. This programme is funded by the Cecil King Memorial Foundation and is aimed at early career mathematicians in the UK or the Republic of Ireland who are registered for a doctoral degree or who have completed their PhD in the 12 months prior to the deadline. The Cecil King Scholarship provides financial support of £6,000 for a study or research visit abroad for a period of three months that opens opportunities for professional development, networking and academic career progression.

The selection process is highly competitive and is based on the quality and merit of the research project, and considers the applicant's motivation, passion, and commitment to their chosen field of study or research.

One Scholarship is usually awarded to a mathematician in any area of mathematics and one to a mathematician whose research is applied in a discipline other than mathematics.

The deadline to submit online applications is 15 November 2023 and shortlisted applicants will be invited to interview in January 2024. For further details visit bit.ly/3rJkiEq.

LMS Mathematical Symposia 2026 – 2030: Call for Proposals

Deadline: 1 December 2023

UK-based institutions or consortia are invited to submit proposals to host the LMS Mathematical Symposia from 2026 to 2030. Proposals should be submitted by 1 December 2023 to grants@lms.ac.uk.

The LMS Mathematical Symposia (currently the LMS–Bath Symposia) are established and recognised international research meetings. Since their founding in 1974, they have provided excellent opportunities to explore an area of research in depth, learn of new developments, and instigate links between different branches.

The Symposia offer opportunities to increase and celebrate equality, diversity, inclusivity, accessibility and sustainability within the mathematical sciences community. To ensure that all UK institutions have the opportunity to benefit from hosting the LMS Mathematical Symposia, the Society encourages applications from universities or consortia of universities that have not traditionally hosted such events.

For further details, see bit.ly/math-symp-2026.

LMS–Bath Mathematical Symposia 2025: Call for Proposals

Deadline: 15 December 2023

Proposals are invited for the LMS–Bath Mathematical Symposia, which will be held at the University of Bath in 2025. Funding is available for two symposia at approximately £40,000 per event.

The format is designed to allow substantial time for interaction and research. The meetings are by invitation only and will be held in July/August, with up to 50 participants, roughly half of whom will come from the UK. A novel element of the symposia is that they will be complemented by a summer school to prepare young researchers such as PhD students, or a "research incubator", where problem(s) related to the topic of the conference is studied in groups. These entire events, summer school/incubator and workshop, will typically last around two weeks.

Prospective organisers should send a formal proposal to the Grants Team (grants@lms.ac.uk) by 15 December 2023. Proposals are approved by the Society's Research Grants Committee after consideration of referees' reports.

Proposals should include:

- A full list of proposed participants, divided into specific categories:
  - Category A — Scientific Organisers
  - Category B — Key Overseas Participants
  - Category C — Key UK-based Participants
  - Category D — Important Overseas Participants
  - Category E — Important UK-based Participants
- Proposers are encouraged to actively seek to include women speakers and speakers from ethnic minorities or explain why this is not possible or appropriate.
- A detailed scientific case for the symposium, which shows the topic is active and gives reasons why UK mathematics would benefit from a symposium on the proposed dates.
- Details of additional support from other funding bodies, or proposed avenues of available funding.
- Indicative plans for the summer school or research incubator.
- Where appropriate, prospective organisers should consider the possibility of an 'industry day'.

For further details, see the Society's website: lms.ac.uk/events/mathematical-symposia or the LMS–Bath symposia's website: bathsymposium.ac.uk.

Before submitting, organisers are welcome to discuss informally their ideas with the Chair of the Research Grants Committee, Professor Andrew Dancer (grants@lms.ac.uk).

European Mathematical Society Prizes: Call for Nominations

Deadline: 31 December 2023

EMS Felix Klein Prize

Nowadays, mathematics often plays the decisive role in finding solutions to numerous technical, economical and organisational problems. To encourage such solutions and to reward exceptional research in the area of applied mathematics, the EMS established the Felix Klein Prize in 1999. The prize is awarded to a scientist, or a group of at most three scientists, under the age of 38 for using sophisticated methods to give an outstanding solution, which meets with the complete satisfaction of industry, to a concrete and difficult industrial problem. Nominations for the prize should be submitted electronically to the chair of the Prize Committee, Professor Peregrina Quintela Estévez (University of Santiago de Compostela; peregrina.quintela@usc.es) with the EMS Office in cc (ems-office@helsinki.fi).

EMS Otto Neugebauer Prize for the History of Mathematics

The prize is to be awarded for highly original and influential work in the field of history of mathematics that enhances our understanding of either the development of mathematics or a particular mathematical subject in any period and in any geographical region. The prize may be shared by two or more researchers if the work justifying it is the fruit of collaboration between them. For the purposes of the prize, history of mathematics is to be understood in a very broad sense. It reaches from the study of mathematics in ancient civilisations to the development of modern branches of mathematical research, and it embraces mathematics wherever it has been studied in the world. In terms of the Mathematics Subject Classification it covers the whole spectrum of item 01Axx (History of mathematics and mathematicians). Similarly, there are no geographical restrictions on the origin or place of work of the prize recipient. All methodological approaches to the subject are acceptable.

The right to nominate one or several laureates is open to anyone. Nominations are confidential; a nomination should not be made known to the nominee(s). Self-nominations are not acceptable. It is the responsibility of the nominator to provide all relevant information to the Prize Committee, including a CV and a description of the candidate's work motivating the nomination, together with names of specialists who may be contacted.

The award comprises a certificate including the citation and a cash prize of €5,000. The Prize will be presented at the Ninth European Congress of Mathematics in Seville, 15-19 July 2024, by the President of the European Mathematical Society. The recipients will be invited to present their work at the congress. The money for the prize fund is offered by Springer-Verlag GmbH.

Nominations for the prize should be submitted electronically to the Chair of the Prize Committee, Professor Karine Chemla (Université Paris Cité – CNRS) chemla@univ-paris-diderot.fr with the EMS Office in cc ems-office@helsinki.fi.

LMS Early Career Fellowships 2023/24: Call for Proposals

Deadline: 14 January 2024

The LMS invites applications from early career mathematicians who have recently or will shortly receive their PhD to apply for the LMS Early Career Fellowship. The aim of this scheme is to provide financial support at the transition period between completing PhD and securing a first postdoctoral position, during which early career mathematicians are encouraged to establish new research links and to expand their network collaborations outside the institution where they completed their PhD studies.

Applicants may submit a proposal for a focused research visit to a UK or overseas institution(s) or for a programme of collaborative in-person visits across the duration of the fellowship outside their home institution. The case should show how the applicant will benefit from professional development and should demonstrate opportunitues to build connections with other researchers to establish and develop a strong academic network.

How you will be supported

The LMS will offer fellowships of between 3 and 6 months and the award will be calculated at £1,552 per month plus £800 allowance for relocation/collaboration visits.

The LMS Early Career Fellowship are partially supported by Heilbronn Institute for Mathematical Research through the UKRI/EPSRC Additional Funding Programme for Mathematical Sciences.

For further details, including testimonials from previous fellows and how to apply, see lms.ac.uk/grants/lms-early-career-fellowships or contact fellowships@lms.ac.uk.

Atiyah UK–Lebanon Fellowships: Call for Applications

Deadline: 31 January 2024

The LMS is pleased to announce the opening of the 2023-24 round of applications for the Atiyah UK–Lebanon Fellowships. It invites applications from established UK based mathematicians who wish to visit Lebanon, and from mathematicians from Lebanon, at the level of an advanced MSc or PhD student and above, who wish to visit the UK.

Since the scheme was launched in 2020 to honour the memory of Sir Michael Atiyah (1929-2019), the Atiyah UK-Lebanon Fellowships have been fostering exchange and interaction between mathematicians in the two countries by supporting a two-way visiting programme. The scheme operates in partnership with the Centre for Advanced Mathematical Sciences (CAMS) at the American University of Beirut.

The Atiyah UK-Lebanon Fellowship will support an academic visit from the UK to Lebanon for a period of between one week and 6 months, or a visit for study or research to the UK from Lebanon for a period of up to 12 months. The maximum funding of £8,600 will cover travel and related costs up to £2,000 and subsistence expenses up to a maximum of £2,200 per month from an allocated maximum of £6,600.

For an academic visit to Lebanon, CAMS will cover accommodation and provision of office space and logistical support. This will be independent of the host institution. There is the possibility of additional subsistence/payment for agreed teaching and some consideration may be given for additional support to Fellows travelling with a family.

There is a possibility for Fellows from Lebanon to apply for further funding from the International Centre for Mathematical Sciences in Edinburgh. Please visit https://www.icms.org.uk/visiting-fellows for more information. Additional support will be available for PhD or MSc candidates in either mathematics or mathematical physics.

For further information about the Fellowships and information on how to apply, please visit bit.ly/3OBoZco.

Undergraduate Research Bursaries

Deadline: 1 February 2024

The Undergraduate Research Bursary scheme provides an opportunity for students in their intermediate years to explore the potential of becoming a researcher. The award provides support to a student undertaking a 6-8 week research project over Summer 2024, under the direction of a project supervisor.

Students must be registered at a UK institution for the majority of their undergraduate degree and may only take up the award during the summer vacation between the intermediate years of their course. Students in the final year of their degree intending to undertake a taught Masters degree immediately following their undergraduate degree may also apply. Applications must be made by the project supervisor on behalf of the student.

For further information contact Lucy Covington (urb@lms.ac.uk). Applications will open in November 2023.

LMS Council Diary — A Personal View

Council met in De Morgan House on the morning of Friday 30 June 2023, with most members present in person and others joining remotely via videoconference. The meeting began with the President noting with great sadness that Vicky Neale, who had been very active in the LMS, had recently passed away. Other items of the President's business included having attended meetings with MPs organised by the Protect Pure Maths Campaign and the President's dinner of the Royal Statistical Society, and that there had been several activities around the Maths for 18 Expert Advisory Group.

The next substantive item of business was a proposal to form an advisory group to support Mathematics Departments facing challenges of various types, of which there have been several in recent years. Council agreed that such a group, possibly titled the University Mathematics Support Group, would be very useful resource for the community and should be set up.

After the Executive Secretary had presented a Strategy Development Update, building on the lively debate that Council had enjoyed at its recent Strategic retreat, Vice-President Gordon gave an update on the work of the LMS Academy Working Group, which is engaging with the various different workstreams of the proto-Academy for Mathematical Sciences.

It was confirmed that there would be an Extraordinary Meeting of Council with the LMS Ambassador for the proto-Academy in early September.

Other business included the Publications Secretary updating Council on the new structure of the joint editorial board of the LMS Bulletin and Journal, and the Treasurer reporting a third quarter review predicting a better-than-expected year-end position and presenting the proposed budget for 23/24. There was also a detailed discussion on the Society's Reserve Fund Policy, the result of which was that a further designated reserve fund of £100K would be added to support LMS global engagement.

We also heard updates on LMS membership and the Mentoring African Research in Mathematics (MARM) Scheme, discussed various committee memberships, and heard a report on the Levelling Up Scheme.

The meeting concluded with the President thanking everyone for their contributions. Members of Council then relocated to Mary Ward House for the LMS General Meeting followed by the lecture by Professor Sir Roger Penrose and the Hardy Lecture 2023, given by Professor Eva Miranda.

Elaine Crooks
Member-At-Large

Report: LMS Regional Meeting and Workshop — Ergodic Theory of Group Extensions

The *Ergodic Theory of Group Extensions* workshop held from 27-29 March 2023, hosted by the University of Warwick, opened with an LMS Regional Meeting, with three speakers who had in common the beautiful and classical topic of the geodesic flow for a hyperbolic surface. Hyperbolic surfaces have been intensely studied for their connections to number theory, spectral theory, algebraic structure, and of course for their geometry. In this context one has fairly explicit counting asymptotics for collections of curves, ranging (chronologically) from Huber's contribution to the exponential growth of all closed geodesics to the polynomial regime of simple closed counting celebrated by those such as Mirzakhani. In the first talk Viveka Erlandsson gave us this historical treatment before telling us of new work with Souto that counts closed geodesics whose type is of a fixed commutator length. Historically, counting closed geodesics for hyperbolic surfaces has a spectral approach. In the more general context of strict negative curvature one can appeal to the hypothesis of hyperbolic dynamics, which loosely means it is chaotic and its behaviour is quasi-random. Mark Pollicott then told us how we can also deal with the larger geometric class of surfaces without conjugate points, in joint work with War. On the topic of spectral properties of hyperbolic surfaces, Michael Magee told us of the problem of optimal spectral gaps and how joint work with Hide exploits randomness to exhibit optimal spectral gaps.

The opening meeting set us up very nicely for the evening reception and the workshop following in the next few days. Away from the beautiful model of the geodesic flow, with all its defining symmetries, what are the nice structures we can look for in our dynamics? The speakers and participants came from a range within this dynamics community, and participants greatly profited from this chance to gather together, learn, and discuss.

Given a dynamical systems with no a priori symmetries of the dynamics, we can put ourselves into the context of the meeting by forming a dynamical extension with a group. The new dynamics are given by the skew product of the base dynamical system with a group acting on itself. Dalia Terhesiu talked to us about strong mixing for certain abelian extensions of hyperbolic dynamical systems with a view to the Lorentz gas, explaining the dynamical challenges in this case. One also looks for qualitative results where the symmetry group is larger and less structured. Manuel Stadlbauer spoke about the Martin boundary, and Rhiannon Dougall about the viewpoint of unitary representations; the former emphasising the context of coarse negative curvature, the latter emphasising countable Markov shifts.

In between the scheduled talks there was time for making valuable use of the chalkboards of the Warwick mathematics department, and for discussion during lunch excursions to the fairly new (and in my humble opinion quite good) dining venues on campus.

A strength of the workshop was that it provided a wider view of the dynamics, tools, and structure than participants may be used to thinking about. Recent trends emerging from the thermodynamic formalism were given by Ian Morris (self-affine sets), Sabrina Kombrink (renewal theory with a view to fractal sets) and Mike Todd (almost Anosov flows). Anders Karlsson told us about the geometry of machine learning, connecting with his earlier work regarding ergodic theorems for cocycles via the geometry of the space. Maik Gröger talked about a problem of unique ergodicity where we generalise away from a single iterated map to an action of a more general group. Finally, in the context of structured spaces, Timothée Bénard and Çarı Sert each talked about the problem of stationary measures for random walks on certain semi-simple Lie groups.

Rhiannon Dougall
Durham University

Report: Hardy Lecture

The Hardy Lecture, given this year by Professor Eva Miranda, focused on the study of the existence of action-angle coordinates in singular symplectic and Poisson manifolds, with a particular emphasis on toric and almost toric actions on singular symplectic manifolds.

The concept of action-angle coordinates plays a crucial role in understanding the dynamics of Hamiltonian systems and has been well established in the context of smooth symplectic manifolds. However, extending these coordinates to singular symplectic and Poisson manifolds presents unique challenges that have attracted significant interest in recent research.

Miranda presented singular symplectic manifolds as a toy model to address problems in Poisson Geometry. She provided a list of motivating examples from different facets of mathematics and discussed the concept of singular symplectic manifolds from b-manifolds to E-symplectic manifolds, as well as introducing the audience to various approaches and techniques employed in the pursuit of action-angle coordinates for singular symplectic manifolds. Some of these methods involve resolution of singularities, symplectic reduction, and the use of symplectic cutting techniques.

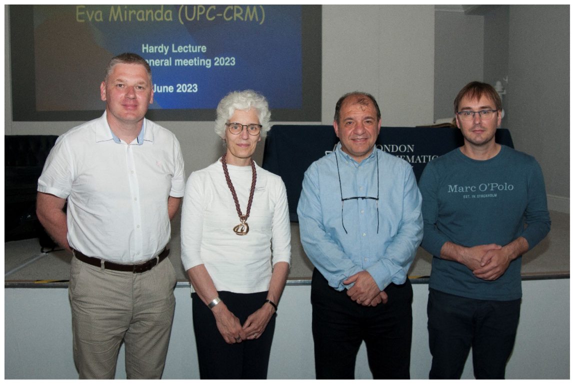
Miranda then delved into the intricacies of toric actions on singular symplectic manifolds. She explained how the torus acts on the manifold, providing a natural framework to study the system's dynamics in the presence of singularities. The interplay between toric actions and action-angle coordinates offered insights into the underlying geometry and topology of the manifold. The study of these mathematical structures is of significant interest in symplectic geometry, algebraic geometry, and Hamiltonian mechanics, and it offers promising avenues for further exploration and research. The lecture not only highlighted the current state of the field but also inspired the audience to explore further this rich and evolving area of mathematics.

The talk was broadcasted via Microsoft Teams. Afterwards, there was a discussion with questions about some particular examples showing these singularities such as the space of geodesics on a pseudo-Riemannian manifold. There was also a question related to Hardy's quote on mathematics being unveiled or discovered. After the official questions, there was a long discussion between the host and the lecturer about the importance of nondegenerate singularities in the context of Poisson Geometry. Several examples considered in the literature by Monnier and Dufour came into play discussing them as E-manifolds. The discussion opened a new path for collaboration between the host and the lecturer.

The day concluded with an official dinner with the host and several invitees.

Alexey Bolsinov
Loughborough University

Report: LMS General Meeting



My name is Volodymyr Proshkin; I participate in the "Solidarity for Mathematicians" program (for more details see newton.ac.uk/event/slm/) from Ukraine. I received an invitation to attend the General Meeting of the London Mathematical Society, which was held in London on June 30. It was my second participation in the meeting. I first attended a meeting at the University of Warwick this spring. I will never forget when I signed the book of the society, where there are autographs of world-famous mathematicians.

Meetings of the London Mathematical Society are a unique opportunity to meet and communicate with researchers from all over Great Britain and listen to exciting lectures. I really liked the research material presented by Professor Eva Miranda (UPC, Barcelona) and Professor Sir Roger Penrose (University of Oxford). I hope to continue to attend LMS meetings! It is practical, exciting and scientific!

Volodymyr Proshkin

Report: LMS Education Day 2023 — Diversifying the Undergraduate Mathematics Curriculum

Report from Mark Hodds (Coventry University):

The 2023 LMS Education Day was held at De Morgan House on Wednesday 24 May. The event focused on discussions and presentations surrounding diversifying the undergraduate mathematics curriculum. The morning began with Professor Duncan Lawson MBE (Coventry University) providing recent updates to the subject benchmark statements in Mathematics, Statistics and Operational Research, of which Duncan was chair of the advisory group. The document can be found at tinyurl.com/3f68sk9k.

The second talk came from Richard Crossman and Emmanuel Ogundimu (Durham University), who focused on their work on decolonising the curriculum. They invited group discussions on various aspects, including what we thought decolonising the curriculum meant, what 'quick wins' we could achieve, and how diverse mathematicians were being incorporated into our teaching. The discussions brought up some interesting points, particularly what decolonising the curriculum means to someone who is not a British native.

The final talk of the morning was given by June Barrow-Green and Brigitte Stenhouse (Open University) where they introduced an open-access resource to help contextualise the curriculum through the history of mathematics. They provided the reasoning for contextualising the curriculum, once again starting a discussion surrounding decolonising the curriculum and its meaning, before introducing the resource where mathematical documents can be searched for according to various factors (e.g. topic, geographical area, language etc.) which would help in teaching and showing how diverse mathematics is as a subject.

Report from Ruth Reynolds (University College London):

In the afternoon we had a broad range of talks. We began with two updates on education initiatives which are being run by the LMS. Professor Jane White (Bath University) reported on TeMaC: Teaching Maths as a Career. The TeMaC initiative was developed to support university mathematics departments in encouraging their students (both undergraduate and postgraduate) to consider a career teaching mathematics, with the aim of combating the lack of suitably qualified subject specialists teaching mathematics in schools. Following this, Jennifer Gunn updated us on the progress of the Levelling Up: Maths programme. Levelling Up: Maths is an access and enrichment programme, administered by the LMS and the IMA, for students from underrepresented backgrounds studying A-level Maths. Following a successful pilot in 2022, there are now 20 universities signed up to the scheme.

Professor Noel-Ann Bradshaw and Professor Tony Mann (University of Greenwich) then presented their talk on students' obstacles to 'belonging' to the mathematics community. In this talk they particularly highlighted some of their students from the University of Greenwich who came to their mathematical study through less traditional paths and have gone on to successful postgraduate study. They emphasised how we should be supporting such students, and we discussed pathways to bring people back into mathematics later in life.

Finally, we finished the day with a talk by Dr Ben Davies (University of Southampton) on STACK: a computer-aided assessment system. STACK is an online assessment system for mathematics and science, created by Chris Sangwin from the University of Edinburgh, which accepts mathematical answers and provides sophisticated feedback. In Ben's talk, he described how STACK was used in UCL to modernise the assessment provision especially with the rapid increase in student numbers during the pandemic. In particular, he described how STACK allows creation of a variety of types of questions such as example generation or step-by-step equivalence reasoning, which are very important for students to answer as part of their formative assessments but very time-consuming to evaluate.

Records of Proceedings at LMS Meetings

General Meeting & Hardy Lecture 2023

The meeting was held on 30 June 2023 at Mary Ward House, Tavistock Square, London, Hardy Lecture 2023. Over 250 members and guests were present for all or part of the meeting, either in-person or online.

The meeting began at 3.30 pm with The President, Professor Ulrike Tillmann FRS, in the Chair.

There were 59 members elected to Membership at this Society Meeting.

14 members signed the Members' Book and were admitted to the Society.

The President, Professor Ulrike Tillmann FRS introduced the first lecture given by Professor Sir Roger Penrose FRS (University of Oxford) on *Non-computability in Physics?*

After tea, The President, Professor Ulrike Tillmann FRS introduced the second lecture by Professor Eva Miranda (UPC-Barcelona) on *From Alan Turing to Fluid Computers: Explored and Unexplored Paths*.

Deputising for the President, the Education Secretary, Dr Kevin Houston, thanked the speakers for their excellent lectures.

Afterwards, a wine reception was held at Mary Ward House. The Society dinner was held at the Wendy House, Bedford Hotel Southampton Row, London.

Projective Geometry and Tensor Spaces

LUCA CHIANTINI

Multilinear objects called tensors have an increasing role in applications. In the last few decades methods of projective geometry have been introduced as a fundamental tool for the investigation of tensor spaces. I will present, in very introductory language, why projective geometry impacts on the study of tensors, some recent achievements, and some future directions of research.

Introduction

At a very early stage of their mathematical studies students meet matrices as useful tools for the investigation of linear spaces. Matrices are almost ubiquitous in any basic aspects of mathematics. The great value of matrices originates from their natural adaptability to encode relations between pairs of sets. Besides their use as data stores, spaces of matrices have a deep algebraic structure. Representation theory measures algebraic structures by comparing them with subalgebras of a matrix algebra.

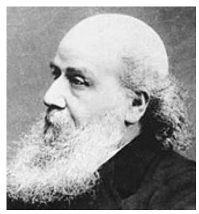
Since matrices are tables in which every entry corresponds to a couple of indices, the most imaginative students quickly flash on the existence of similar, multidimensional, objects, in which the number of indices grows indefinitely. Despite the fact that multidimensional arrays can be defined and handled quite naturally in most programming languages, systematic studies of multidimensional matrices are often missing even in advanced programs in mathematics, or they are relegated to advanced courses. A reason is due to the fact that while we have many deep tools to explore spaces of ordinary matrices, on the other hand we hardly know how to determine even basic properties of multidimensional matrices.

Multidimensional matrices (or tensors, as I will call them here) are a natural way in which one can represent relations between three or more sets. Thus, in the development of modern theories which require one to trace the interaction of several discrete variables, the role of tensors acquires a growing importance. This encouraged recent studies on tensor spaces, stimulated by possible applications to Quantum Physics, Information Theory, Signal Processing, Artificial Intelligence, Statistics, and more.

In the last three decades a new geometric perspective has been developed in the theory of spaces of tensors, based on the investigation of some algebraic varieties in projective spaces. It was classically known that the most relevant varieties of tensors are algebraic or quasi-algebraic. Only recently, however, has the use of advanced projective techniques proved its effectiveness in the production of results on tensors. This initiated a flow of studies in which sophisticated geometric tools are employed in the analysis of tensor spaces.

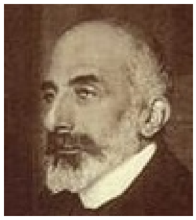
The goal of this note is a description of the status of the research on special projective varieties (Segre and Veronese varieties) and their subvarieties (e.g. configurations of points), and how these objects give us important information about the structure of general, or specific, tensors.

Short historical note



J. J. Sylvester

A systematic theory of multilinear objects appears in the studies of A. Cayley and J. J. Sylvester, in the middle of the XIX century. In the second part of the same century an initial theory of multidimensional matrices was developed. Originally the term ‘tensor’ denoted multilinear maps between vector spaces. As multilinear maps, tensors played a basic role in the differential geometry that supports the general relativity theory. After a choice of basis, there is a natural one-to-one correspondence between multilinear maps and multidimensional matrices. Thus the term ‘tensor’ is actually employed to denote multidimensional matrices as well.



C. Segre

Some fundamental subvarieties in a space of tensors, which generalizes sets of matrices of rank 1, correspond to multilinear images of products of spaces. These subvarieties were systematically introduced by C. Segre, in the second part of the XIX century, as embeddings of products of projective spaces in a wider ambient space. So, the varieties are universally known as 'Segre varieties'.

G. Veronese

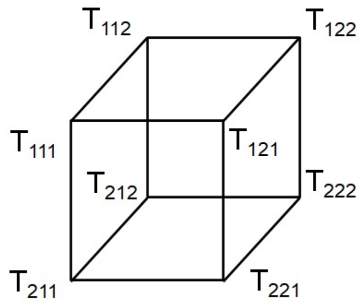
At the same time G. Veronese introduced the symmetric analogue of Segre varieties, now called 'Veronese varieties'. Though intimately related, the study of algebraic properties of tensors and the geometric study of Segre and Veronese varieties proceeded almost separately for a long time. In the 1970s, basic results by J. Kruskal and by V. Strassen suggested that the introduction of geometric tools could yield significant advances in the investigation of tensor spaces. The milestone that convinced researchers of the usefulness of the interplay between projective geometry and tensor analysis is the celebrated result, proved in the middle of the 1990s by J. Alexander and A. Hirschowitz, that computes the value of the rank of general symmetric tensors of any given shape.

Spaces of tensors

I will use the term 'tensor' to indicate a multiindexed table of entries. Entries lie in a fixed base field $\mathbb{F}$. Usually $\mathbb{F}$ is the field of real numbers or the field of complex numbers, even though recent studies on tensors defined over finite fields have been developed. I will use the notation

$$T = (T_{i_1,\dots,i_k})$$

to denote the entries of the tensor T. Every index i_j ranges in a discrete interval $i_j = 1,\dots,n_j$. Thus I will say that T above is a tensor of type $n_1\times\cdots\times n_k$. Usual matrices are tensors with two indices. Vectors can also be considered as tensors, with only one index. It is immediate to see that entry-by-entry sum and product by a scalar define the structure of a linear space on the set of tensors of fixed type.



A $2\times 2\times 2$ tensor

There is a basic operation on tensors: the tensor product $\otimes$, defined as follows. The tensor product $S = T\otimes U$ of $T = [T_{i_1,\dots,i_k}]$ and $U = [U_{j_1,\dots,j_m}]$ has entries

$$S_{i_1,\dots,i_k,j_1,\dots,j_m} = T_{i_1,\dots,i_k}\cdot U_{j_1,\dots,j_m}.$$

When $T\in\mathbb{F}^n$ and $U\in\mathbb{F}^s$ are vectors, then $T\otimes U$ is the $n\times s$ matrix obtained as the product of the column T times the row U.

The standard notation for the space of tensors of type $n_1\times\cdots\times n_k$ is

$$\mathbb{F}^{n_1}\otimes\cdots\otimes\mathbb{F}^{n_k}.$$

The notation can be misleading. Certainly for any choice of vectors $v_1\in\mathbb{F}^{n_1},\dots,v_k\in\mathbb{F}^{n_k}$ the (iterated) product $v_1\otimes\cdots\otimes v_k$ is an element of $\mathbb{F}^{n_1}\otimes\cdots\otimes\mathbb{F}^{n_k}$, but general tensors cannot be written as a single product. For instance, the product of two vectors can only produce matrices of rank 1. Instead, the generic tensor T of type $n_1,\times\cdots\times n_k$ is a finite sum

$$T = \sum_{i=1}^{r} v_{i1}\otimes\cdots\otimes v_{ik}. \qquad (1)$$

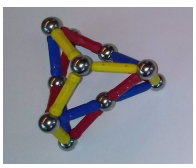
where each summand belongs to $\mathbb{F}^{n_j}$. The way in which one writes T as a sum of products of vectors is a fundamental point of multilinear algebra. Despite the fact that in the case of matrices we have a clear view on which different expressions $\sum_{i=1}^r v_{i1}\otimes v_{2i}$ can describe the same object, no complete description is known for general multiindexed tensors. A couple of examples can clarify the matter.

Let T be the $2\times 2\times 2$ tensor that records the probability of obtaining a result by throwing three distinct coins. One can put 1 = head and 2 = tail and define T_{ijk} accordingly, so that T_{111} is the probability to obtain head-head-head, and so on. Every coin can be loaded, so that it can give a head with probability different from $1/2$. If one indicates with (p_1^i, p_2^i) the probabilities that the i-th coin returns (head, tail), then T is equal to the product $(p_1^1, p_2^1) \otimes (p_1^2, p_2^2) \otimes (p_1^3, p_2^3)$ exactly when the three coins behave independently, for the condition is equivalent to have $T_{ijk} = p_i^1 p_j^2 p_k^3$ for all i, j, k. On the other hand, if there exists some mechanism for which the i-th coin is influenced by the other two coins, then T is no longer a single product. It is given by a more complicated expression which depends on the way the three coins influence each other.

There are tensors T^n with entries 0 or 1 which encode the row-by-column multiplication of $n \times n$ matrices. T^n can be roughly constructed as follows: if A, B, C are $n \times n$ matrices with $C = AB$, then considering A, B, C as vectors in $\mathbb{F}^{n^2}$ one defines

$$T^n_{ijk} = \text{ coefficient of } A_i B_j \text{ in the entry } C_k$$

Thus T^2 is a $4 \times 4 \times 4$ tensor, T^3 is a $9 \times 9 \times 9$ tensor, etc. From this point of view, the tensor product of A, B, C represents an operation in which one linear combination of the entries of C corresponds to a single multiplication of a linear combination of entries of A times a linear combination of entries of B. Hence, the expression of T^n as a sum of products corresponds to algorithm that through a series of multiplications of (linear combinations of) entries of A and B computes the product AB. The number of summands indicates the number of multiplications needed in the process. In complexity theory sums and multiplication by a fixed coefficient are considered as negligible, in terms of time or memory consuming, with respect to the multiplication of two variables. So, the number of summands of an expression of T^n is a reasonable measure of the intrinsic complexity of the algorithm. Everybody knows how to multiply 2×2 matrices with 8 products. It corresponds to an expression of T^2 with 8 summands. For a long time this was considered as the best possible decomposition of T as a sum of products.

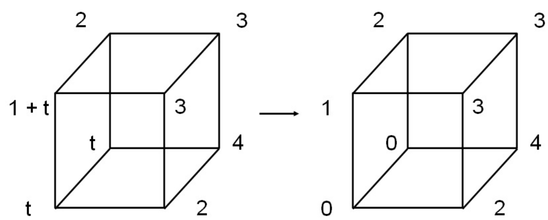


Strassen algorithm constructed!

With great general surprise, in 1969 V. Strassen found a decomposition with only 7 summands. The new algorithm is now known to be optimal, and effectively implemented in some computer packages for linear algebra. So, attention turned to compute the complexity of the product of 3×3 and larger matrices. With no success. At the time of writing, at the beginning of 2023, we only know that the complexity of T^3 is at least 19 and at most 23, but we do not know the exact value. Since T^3 is a $9 \times 9 \times 9$ tensor, with 729 entries, you may think that a big computer could exploit all the possibilities and provide the final answer. Go ahead and try it! The most powerful computers nowadays cannot perform all the calculations needed to compute the complexity of T^3. So, take it as an opportunity, and not as a sign of failure: humanity does not know actually the true complexity of the product of two 3×3 matrices.

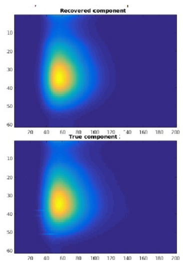
Going back to the general setting, in any space of tensors of given type $n_1 \times \cdots \times n_k$ one finds the set $\mathbb{X}$ of products $v_1 \otimes \cdots \otimes v_k$ of vectors. $\mathbb{X}$ is a non-linear object. It contains tensors that represent 'separable states' in quantum physics. Following the language of matrices, tensors $T \in \mathbb{X}$, corresponding to single products, are called 'tensors of rank 1'. Then one defines the rank of any tensor T as the minimal number of summands of an expression of T as sum of elements of $\mathbb{X}$, as in (1). As explained in the example of the matrix multiplication tensors, the rank can be considered as a measure of the complexity of the tensor T. It is important to stress that no general method is known to determine the complexity of a multiindexed tensor. Indeed the problem of determining the expression (1) for a general tensor is known to be NP-hard. Only specific methods are available, and only for few special cases.

One must observe that in a continuous 1-dimensional algebraic family of matrices $\{M_t\}$ the rank is semi-continuous, in the sense that there is a generic value of the rank that holds for all matrices except for a finite number of cases. The rank of exceptional matrices can only be smaller than the generic rank. On the contrary, there are algebraic families of tensors whose special elements have rank bigger than the generic one. An example can be found in the image below. Notice that the rank of a multiindexed tensor T in general is not determined by the space of vectors obtained by slicing T in any possible direction.



Family of rank 2 tensors with a limit of rank 3

Another fundamental difference between matrices and higher multiindexed tensors concerns the number of different minimal expressions for a general T. Here 'different' means except trivialities, like reordering or scalar multiplication. In the case of matrices there are infinitely many minimal expressions (1) for any matrix of rank greater than 1. Any basis for the space of rows or the space of columns of a matrix can be used to obtain the expression. On the contrary, it is known that tensors with three or more indices often have a unique minimal decomposition. The consequence can be important for applications.



Marker image reconstruction

For example, assume that a researcher wants to detect the structure of a biological system by means of liquid markers. Markers particularly useful to this goal are fluorophores, molecules that react to a beam of light with an emission with a specific wavelength. The intensity x_{ij} of the light that is emitted at the j-th wavelength when a fluorophore is excited at the i-th wavelength obeys:

$$S_{ij} = \chi \lambda_i \mu_j,$$

where λ_i is the fraction of light absorbed at wavelength i, μ_j is the fraction of light emitted at wavelength j, and χ is a chemical constant proportional to the concentration of the fluorophore. When r fluorophores occur in a sufficiently diluted solution, the total emitted light satisfies:

$$S_{ij} = \sum_{q=1}^{r} \chi_q \lambda_{iq} \mu_{jq}.$$

Consequently, when analyzing k sample solutions concurrently, the emitted light at the j-th wavelength when the k-th solution is excited by light at the i-th wavelength satisfies:

$$T_{ijk} = \sum_{q=1}^{r} \chi_{qk} \lambda_{iq} \mu_{jq} \quad \text{i.e.} \quad T = \sum_{q=1}^{r} \chi \otimes \lambda \otimes \mu.$$

The researcher knows the vectors λ, μ of the fluorophores injected in the system, and measureS the final tensor T. When T is general, the uniqueness of the expression enables to recover the concentration profiles of the mixtures of markers. Observe that the profiles cannot be recovered by a single solution, because the expression of a matrix S_{ij} as product is not unique.

From a mathematical point of view, a series of questions concerning the complexity of tensors T can thus be articulated as follows:

Q1 What is the rank (complexity) of T?

Q2 How many minimal expression of T are there? In particular, is the minimal expression of T unique?

Q3 Find the summands of a minimal expression of T.

The intermediate question Q2 is relevant also because heuristic iterative methods to write T as a sum of products are more reliable when the solution is unique. In this case T is said to be 'identifiable'. The solutions to questions Q1, Q2, Q3 are unknown in general. One must consider that the problems are relatively easier when the tensor T is sufficiently generic (even though it is not completely clear what 'generic' means in this contest), while in specific cases the answer can be much harder. For instance, it is known that a minimal expression of a tensor of matrix multiplication T^n is not unique, because it is subject to the action of linear group $\mathcal{G}$ on rows and columns of the matrices. In this case question Q2 can be rephrased by asking if all minimal expressions belong to the same orbit of $\mathcal{G}$.

The projective point of view

The structure of spaces of tensors is deeply related to the geometry of some algebraic varieties in projective spaces. The relation was classically well known. Yet, only in the last 30 years has a systematic geometric approach produced sensible advances in our knowledge on both sides, projective geometry and tensor theory.

Since many achievements are easier to describe for tensors defined over the complex field, whose algebraic structure has wider completeness, from now on I will focus on the case

$$\mathbb{F} = \mathbb{C}.$$

It remains implicit that specific questions on tensors over the reals, or even over $\mathbb{Q}$ or over finite fields, need further treatment.

If V is any complex linear space, the projective space over V is defined by factoring out the scalar multiplication on V. Technically, one puts on V the equivalence relation $\sim$ that associates vectors v, w when they are proportional, i.e. there exists a scalar $a \in \mathbb{C}$ with $w = av$. In order to obtain an equivalence relation, one must drop the zero vector. Thus the projective space $\mathbb{P}(V)$ corresponds to the quotient $(V \setminus \{0\})/\sim$. Points in $\mathbb{P}(V)$, i.e. equivalence classes in $V \setminus \{0\}$, correspond to linear subspaces of dimension 1. Hence projective dimensions correspond to linear dimensions minus 1. So, a linear space V of dimension $n+1$ defines a projective space $\mathbb{P}^n = \mathbb{P}(V)$. In the sequel, I will denote by $[v]$ the point of $\mathbb{P}(V)$ corresponding to the equivalence class of $v \in V$. If $x_0, \dots, x_n$ are the coordinates of v with respect to a basis of V, I will denote the point $[v]$ also with $[x_0 : x_1 : \cdots : x_n]$. Linear changes of coordinates do not modify the structure of $\mathbb{P}^n$.

Algebraic varieties in $\mathbb{P}^n$ are subsets defined by the vanishing of a set of homogeneous polynomials. One needs 'homogeneous' polynomials to preserve the vanishing in the equivalence class representing a point. It is well known that any algebraic variety can be defined by a finite set of homogeneous polynomials. From the construction it is immediate that projective algebraic varieties correspond to projective cones in the affine space $\mathbb{A}^{n+1}$ associated to V.

For any fixed type $n_1 \times \cdots \times n_k$ of tensors one can consider the projective space $\mathbb{P}(\mathbb{C}^{n_1} \otimes \cdots \otimes \mathbb{C}^{n_k})$. The space has (projective) dimension

$$N = (\Pi_{i=1}^k n_i) - 1.$$

The introduction of projective geometry in the study of tensors is motivated by the observation that most relevant subvarieties of tensors correspond to projective subvarieties of $\mathbb{P}^N$. For one, the set $\mathbb{X}$ of tensors T of rank 1 is defined by homogeneous equations corresponding to 2×2 subminors ot T, including the skew ones. These subminors are defined as follows. Fix any two entries, $T_{i_1,\dots,i_k}$ and $T_{j_1,\dots,j_k}$. Fix any index $u \in \{1, \dots, k\}$. Then the homogeneous polynomial

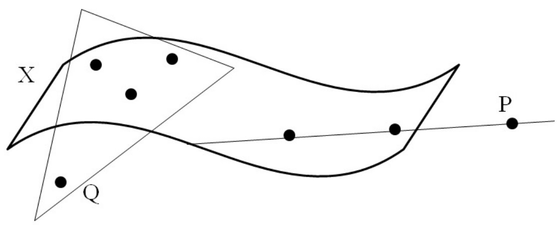
$$T_{i_1,\dots,i_k} T_{j_1,\dots,j_k} - T_{i_1,\dots,j_u,\dots,i_k} T_{j_1,\dots,i_u,\dots,j_k}$$

vanishes on $\mathbb{X}$, and $\mathbb{X}$ is defined exactly by the vanishing of all such polynomials (over $\mathbb{C}$).

One may wonder, what is the reason for using projective spaces, instead of the more familiar notion of linear spaces, in the description of spaces of tensors. The best possible answer is that projective spaces $\mathbb{P}^N$ are complete compact spaces, and this permits the introduction in $\mathbb{P}^N$ of several geometric tools which cannot be used easily in affine spaces. A daunting aspect of projective spaces is the loss of an elementary metric. Since in a projective space one can always multiply coordinates by a scalar, the usual metric becomes completely useless. In a projective space only directions matter. This corresponds to the fact that just turning slightly your telescope when looking at a starry sky, you can switch between two galaxies some billions of light years apart. It is in fact possible to recover a metric in $\mathbb{P}^n$ (the Fubini-Study metric), by substituting angles between directions for the euclidean distance between coordinates. Yet, the lack of a direct measure for distances can discourage people from taking seriously the projective approach to the study of tensors. I will try below to convince the reader that projective techniques can prove themselves useful in the investigation of tensor spaces, and distances are not fundamental at a first analysis.

A deep difference between affine and projective geometry is the fact that products of projective spaces are not projective spaces themselves. Indeed products are not linear objects. In particular $\mathbb{P}^1 \times \mathbb{P}^1$ does not coincide globally with $\mathbb{P}^2$. The reason is that for points in $\mathbb{P}^1 \times \mathbb{P}^1$ one can freely re-scale the first pair and the second pair of coordinates separately, while in $\mathbb{P}^2$ all the coordinates must be scaled together. A relation between products and projective spaces was discovered by C. Segre. One can find an injective map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$ by sending $([x_0 : x_1], [y_0 : y_1])$ to $[x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1]$. Clearly re-scaling the coordinates of each factor of $\mathbb{P}^1 \times \mathbb{P}^1$ determines a total rescaling of the coordinates of the image in $\mathbb{P}^3$, so that the map is well-defined. The map is known as the 'Segre embedding', and can be generalized to any product $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_k}$ by sending $(P_1, \dots, P_k)$ to the point $Q \in \mathbb{P}^N$, with $N = (\Pi_{i=1}^k (m_i+1)) - 1$, in which each coordinate is the product of one coordinate of P_1 times one coordinate of P_2 …times one coordinate of P_k. Thus, if v_i is a vector in $\mathbb{C}^{m_i+1}$ representing coordinates for P_i, then the image of $(P_1, \dots, P_k)$ corresponds to the point of $\mathbb{P}^N$ whose coordinates are given by the tensor product $v_1 \otimes \cdots \otimes v_k$. From this point of view the target space can be identified with a tensor space $\mathbb{P}(\mathbb{C}^{m_1+1} \otimes \cdots \otimes \mathbb{C}^{m_k+1})$ (as usual, dimensions grow by 1 when passing from the projective to the affine notation) and the image of $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_k}$ in the Segre embedding (a projective variety, also called 'Segre variety') corresponds to the set $\mathbb{X}$ of rank 1 tensors.

Having established that rank 1 tensors are points of a Segre variety $\mathbb{X}$, let us see what describes rank 2 tensors. Any tensor T of rank 2 is a sum of two (non-proportional) tensors T_1, T_2 of rank one. More generally, since multiplication by a non-zero scalar is irrelevant to the rank, then any linear combination $aT_1 + bT_2$ with $a, b \in \mathbb{C} \setminus \{0\}$ is also the sum of two rank 1 tensors, so it has rank at most 2. Linear combinations of T_1, T_2 form a linear subspace of dimension 2 in $\mathbb{C}^{N+1}$, hence they correspond to a projective line, spanned by the points $[T_1], [T_2] \in \mathbb{P}^N$. Thus all lines joining two points of the Segre variety $\mathbb{X}$ contain tensors of rank ≤ 2. In projective geometry, the union of lines joining two points of a variety $\mathbb{Y}$ is called the 'strict secant variety' of $\mathbb{Y}$. The term 'strict' refers to the fact that the union of the lines is not, in general, a projective variety itself: there are limits (tangent lines) whose points do not lie in any line spanned by two distinct points of $\mathbb{Y}$. The strict secant variety is however dense in a projective variety, the 'secant variety' of $\mathbb{Y}$, denoted by $\sigma_2(\mathbb{Y})$. Thus tensors of rank 2 are dense in the secant variety of the Segre variety.



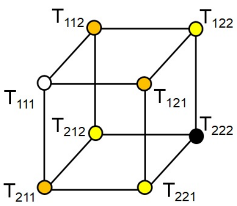
Secant spaces

Secant varieties are classical objects in projective geometry. The main reason for their introduction comes from the study of the minimal space in which one can embed a variety. Namely the projection of $\mathbb{Y} \subset \mathbb{P}^N$ to $\mathbb{P}^{N-1}$ from a vertex P defines an isomorphism on $\mathbb{Y}$ if and only if P sits outside the secant variety $\sigma_2(\mathbb{Y})$. Thus, starting with one embedding of $\mathbb{Y}$ in some projective space $\mathbb{P}^n$, one can produce an embedding in a space of lower dimension by projection, provided that $\sigma_2(\mathbb{Y})$ does not fill the ambient space. There is a straightforward expected dimension for $\sigma_2(\mathbb{Y})$. Lines joining two points of $\mathbb{Y}$ are parameterized by the 2-symmetric product $\mathbb{Y}^{(2)}$ outside the diagonal. Thus the secant variety of $\mathbb{Y}$, of dimension n, is the union of a $2n$-dimensional family of lines. So it has dimension at most $2n + 1$. This is the easy proof that any variety of dimension n in $\mathbb{P}^N$, $N > 2n + 1$, can be embedded, by successive projections, in $\mathbb{P}^{2n+1}$. In $\mathbb{P}^{2n+1}$ the procedure stops since the secant variety is expected to fill the ambient space. There are however cases in which the dimension of $\sigma_2(\mathbb{Y})$ is smaller than $2n + 1$. This happens exactly when a general point of $\sigma_2(\mathbb{Y})$ sits in infinitely many secant lines. There is a large literature on varieties $\mathbb{Y}$ whose secant variety has dimension smaller than expected. The theory culminated with the work of F. Zak, who in 1981 classified smooth varieties for which $\sigma_2(\mathbb{Y})$ has (minimal) dimension $1 + (3/2)n$.

When the rank grows, one can repeat the procedure. The set of tensors of rank $\leq r$ is dense in the r-th secant variety $\sigma_r(\mathbb{X})$ which is the smallest projective variety that parametrizes points of $\mathbb{P}^N$ that can be written as a linear combination of r points of $\mathbb{X}$. The study of general secant varieties is a huge classical topic in projective geometry. Transferring results from the geometry of secant varieties of Segre varieties is the way in which projective geometry impacts on tensor theory. Conversely, tensor theory has suggested problems in secant varieties whose study has yielded advances in projective geometry.

A subclass of tensors which is worthy of special mention in connection with applications of projective geometry is the class of symmetric tensors. A tensor is symmetric if entries obtained by any permutation of the indices are equal. A tensor T of rank 1 is symmetric if and only if T is equal to a product of type $v \otimes \cdots \otimes v$, i.e. T is a power $v^{\otimes k}$ of a single tensor. In the Segre map, the coordinates of the image of $v^{\otimes k}$ are given by all monomials (monic, i.e. with coefficient 1) of degree k in the coordinates of v. Thus, we are applying the Segre map only to elements of the diagonal of $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$, which is clearly isomorphic to $\mathbb{P}^n$. The restriction of the Segre map to the diagonal is called the 'Veronese map' and its image is a 'Veronese variety'. Avoiding repetitions due to the symmetry, the image of the Veronese map sits in a projective space of (projective) dimension $M = m - 1$, where m is the number of monic monomials of degree k in $n + 1$ variables. Any symmetric tensors can be written as a finite sum of symmetric tensors of rank 1. Thus the linear space $Sym^k(\mathbb{C}^{n+1})$ of symmetric tensors of type $(n + 1) \times \cdots \times (n + 1)$ (k times) is the linear span of the set of powers $v^{\otimes k}$.

Projective spaces often classify geometric objects. This special aspect of projective geometry was noted by C. Segre in the late XIX century, leading to a discussion with G. Veronese, who on the contrary supported a totally traditional view of points in $\mathbb{P}^n$ as pure atomic objects. One example of projective spaces as classification spaces is given by plane conics. Proportional equations define the same conic. Hence the relation between a conic Γ and the 6-uples of coefficients of its equations establishes a one-to-one correspondence between conics and points of a projective space $\mathbb{P}^5$. In the correspondence, pencils of conics correspond to lines in $\mathbb{P}^5$. The correspondence can be generalized to homogeneous polynomial (that will be called 'forms' in the sequel) of any degree k in any number $n+1$ of variables, so that tensor classes $[T] \in \mathbb{P}(Sym^k(\mathbb{C}^{n+1}))$ correspond to hypersurfaces of degree k in $\mathbb{P}^n$. In the correspondence rank 1 symmetric tensors can be identified as forms which are k-th powers of linear polynomials. The expression of $T \in Sym^k(\mathbb{C}^{n+1})$ as a sum of symmetric tensors of rank 1 corresponds to the expression of the form associated to T as a sum of k-th powers. The minimal number of summands in an expression $T = \sum_{i=1}^{r}(v_i^{\otimes k})$ is the symmetric rank, also called 'Waring rank', of the form associated to T. The symmetric rank is considered as a good estimate of the complexity of a form.



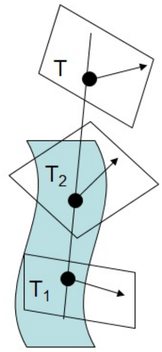
$$\longrightarrow \quad T_{111}x^2 + (T_{112}+T_{121}+T_{211})x^2y + \\ +(T_{122}+T_{212}+T_{221})xy^2 + T_{222}y^2$$

Symmetric $2 \times 2 \times 2$ tensor and associated form

Recent achievements

I am ready now to describe some recent achievements in tensor theory coming from projective geometry.

I will start the story from the result of Alexander and Hirschowitz characterizing the dimension of secant varieties to Veronese varieties (1995, see [2]). For any (irreducible) projective variety $\mathbb{Y} \subset \mathbb{P}^M$ of dimension n the secant variety $\sigma_r(\mathbb{Y})$ has an expected dimension, equal to the smaller of the dimension M of the ambient space and the number of parameters for points in linear spans $\mathbb{P}^{r-1}$ of choices of r points in $\mathbb{Y}$, i.e. $rn + (r-1)$. Alexander and Hirschowitz proved that for Veronese varieties the expected dimension holds, except for a short list of (classically well known) cases. The result immediately translates to the theory of symmetric tensors. We know now the dimension of the variety of symmetric tensors of any symmetric rank r in $\mathbb{P}(Sym^k(\mathbb{C}^{n+1}))$, up to the first value for which the variety is dense in the ambient space. In particular, for all k, n we know the rank of a general symmetric tensor in $\mathbb{P}(Sym^k(\mathbb{C}^{n+1}))$, hence the symmetric rank of a general form of any degree in any number of variables.



Terracini's Lemma

The proof of the Alexander-Hischowitz result is purely geometric. In the early XX century A. Terracini characterized the tangent space Λ to a secant variety $\sigma_r(\mathbb{Y})$ at a generic point u in the span of generic $P_1, \dots, P_r \in \mathbb{Y}$. The result is quite intuitive: Λ is the span of the tangent spaces to $\mathbb{Y}$ at $P_1, \dots, P_r$. In the interpretation of points in $\mathbb{P}^M$ as hypersurfaces of degree k in $\mathbb{P}^n$, Λ corresponds to hypersurfaces with a point of multiplicity at least 2 at the preimages of $P_1, \dots, P_r$ in the Veronese embedding $\mathbb{P}^n \to \mathbb{P}^M$. Thus the problem becomes a question of polynomial interpolation: evaluate the dimension of the space of hypersurfaces with multiple points at r generic points of $\mathbb{P}^n$. Alexander and Hirschowitz solved the problem with a series of specializations of the r points to linear subspaces. When the final, special set of points determines a space of singular forms with the expected dimension, then by semicontinuity the expected dimension holds for general points.

With essentially the same method, adapted through a birational transformation that translates interpolation in a product of projective spaces into interpolation in a single, big projective space, M.V. Catalisano, A.V. Geramita, and A. Gimigliano found a similar result for Segre varieties, images of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ any number of times (2005, see [5]). Thus we know now the dimension of varieties of tensors of type $2 \times \cdots \times 2$ for any value of the rank up to the generic one. No similar complete results are known for other series of space of tensors with growing dimensions. However, we know the dimension of secant varieties for a huge list of special cases.

There is a strong connection between the dimension of secant varieties of a projective variety $\mathbb{Y}$ and the existence of families of special subvarieties of $\mathbb{Y}$. In [6] it is shown that a secant variety $\sigma_r(\mathbb{Y})$

always attains the maximal possible dimension (the expected one) unless for generic points $P_1,\dots,P_r \in \mathbb{Y}$ there exists a subvariety $Z \subset \mathbb{Y}$, containing the points, which spans a projective space of low dimension (a 'degenerate' subvariety). The existence of such subvarieties Z forces the secant spaces to $\mathbb{Y}$ to pack together, so that a general point of $\sigma_r(\mathbb{Y})$ belongs to infinitely many secant spaces. In practice, secant varieties always occupy the maximal possible extent of space, unless $\mathbb{Y}$ contains subvarieties acting as a sort of internal binding that inhibits secant spaces from stretching completely. Thus we have a recipe to determine the dimension of secant varieties in terms of the internal geometry of $\mathbb{Y}$. This is sufficient to determine the dimension of secant varieties in many (but not all) specific cases.

Several computations on secant varieties of projective varieties $\mathbb{Y} \subset \mathbb{P}^N$ follow from an incidence diagram. Consider the set of $(r+1)$-tuples

$$a\sigma_k(\mathbb{Y}) = \{(Q, P_1, \dots P_r) : P_1, \dots, P_r \in \mathbb{Y} \text{ and } Q \in \text{the linear span } \langle P_1, \dots, P_r\rangle\}.$$

The set $a\sigma_r(\mathbb{Y})$ called the 'abstract secant variety', is dense in a projective subvariety of $\mathbb{P}^N \times \cdots \times \mathbb{P}^N$ ($r+1$ times). It is clear that the image of the first projection $s_r : a\sigma_r(\mathbb{Y}) \to \mathbb{P}^N$ defines, with its closure, the secant variety $\sigma_r(\mathbb{Y})$.

$$\begin{array}{ccc} & a\sigma_r(\mathbb{Y}) & \\ \swarrow s_r & & \searrow \\ \sigma_r(\mathbb{Y}) \subset \mathbb{P}^N & & (\mathbb{P}^N)^r \end{array}$$

The advantage of the introduction of $a\sigma_r(\mathbb{Y})$ relies on the standard fact that for any projective map of varieties $f : \mathbb{X} \to \mathbb{Y}$ the 'fibres' of f, i.e. the inverse images of points $P \in \mathbb{Y}$, are subvarieties of $\mathbb{X}$, and when the image of f is dense in $\mathbb{Y}$ then the dimensions are linked by the formula

$$\dim(\mathbb{X}) = \dim(\mathbb{Y}) + \dim(f^{-1}(P))$$

for P general in $\mathbb{Y}$. Since the abstract secant variety is birational to a $\mathbb{P}^{r-1}$-bundle over the symmetric product $\mathbb{Y}^{(r)}$, then its dimension is always equal to $r\dim(\mathbb{Y})+(r-1)$. The computation of the dimension of $\sigma_r(\mathbb{Y})$ is thus reduced to the computation of the dimension of a general fibre of the map s_r.

The study of fibres of the map s_r not only determines the dimension of varieties of tensors of given rank, but it is also useful in determining how many expressions of a general tensor T as a sum of r products one can find. This is exactly question Q2 above. Indeed if $\mathbb{P}^N$ is the space of tensors containing T and $\mathbb{Y} \subset \mathbb{P}^N$ is the corresponding Segre or Veronese variety of tensors of rank 1, then $[T] \in \sigma_r(\mathbb{Y})$, and the fiber of the map s_r over $[T]$ contains the summands of minimal expressions of T as sums of products. Thus T is identifiable if and only if the fiber $s_r^{-1}([T])$ contains a unique r-tuple of points (in any possible order). By using Terracini's description of the tangent space to secant varieties, one can again rephrase the identifiability problem for a general tensor T of given rank to a polynomial interpolation problem in a projective space. With this reduction, a complete description of cases in which a general symmetric tensor of given rank is identifiable is now available (here 'identifiability' is intended with respect to symmetric decompositions). I refer to [7, 9] for details. In the non-symmetric case a complete description of ranks for which generic identifiability holds is known for no types of tensors. In the special case of tensors of type $2\times\cdots\times 2$, thanks to a result of A. Casarotti and M. Mella that links identifiability to the dimension of secant varieties, we know all the values of the rank for which identifiability holds, except for a very narrow range close to the generic rank.

It is worth observing that, as in the case of the dimension of secant varieties, as soon as $\sigma_r(\mathbb{Y})$ does not fill the ambient space, then identifiability holds for a general $[T] \in \sigma_r(\mathbb{Y})$, unless $\mathbb{Y}$ contains special degenerate subvarieties passing through r general points. Thus, even from a numerical point of view, secant varieties σ_r tend to expand themselves to the maximal extension in which a general point sits in a unique r-secant space, unless internal bindings inhibit the maximal stretching.

Up to now I have discussed the situation for generic tensors of given rank. In projective geometry a property holds for 'generic' objects in $\mathbb{Y}$ when there exists an open dense subset of $\mathbb{Y}$ (technically: in the Zariski topology) in which the property holds. So, 'generic tensor' means all tensors outside a proper projective subvariety (which always has measure 0). In most cases however, and identifiability of tensors is one of them, we have no exact characterization of open dense subsets in which the property holds.

Turning to the study of rank and identifiability of a given, specific tensor T, even in the symmetric case our knowledge is much less developed. Projective geometry again provides tools for the investigation of the problem. The basic result goes back to J.B. Kruskal (1977), who proved a criterion to decide if a specific expression

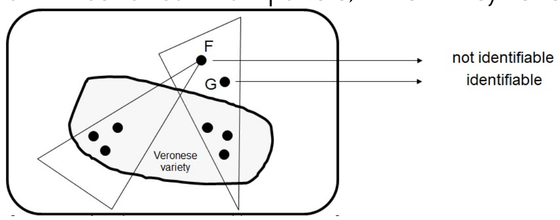
$$T = \sum_{i=1}^{r} v_{i1} \otimes v_{i2} \otimes v_{i3} \qquad (2)$$

of a three-indexed tensor is minimal, so that it computes the rank of T, and is unique (up to trivialities). If T has type $(n_1+1)\times(n_2+1)\times(n_3+1)$ then the expression (2) determines three finite subsets: $Z_1 = \{[v_{11}],\dots,[v_{r1}]\} \subset \mathbb{P}^{n_1}$, and the similar subsets $Z_2 \subset \mathbb{P}^{n_2}$, $Z_3 \subset \mathbb{P}^{n_3}$. For any finite subset $Z \subset \mathbb{P}^n$ the 'Kruskal rank' of Z is the maximal q such that all subsets of cardinality q in Z are linearly independent. Then, denoting by q_i the Kruskal rank of Z_i, the expression (2) is unique when

$$r \leq \frac{q_1+q_2+q_3-2}{2}.$$

There are versions of Kruskal's criterion for multiindexed tensors, and for symmetric tensors. H. Derksen proved that the criterion cannot be improved without adding conditions on the geometry of the sets Z_i (2013). Refined criteria were recently introduced in [8], in terms of the rank of sub-tensors and sub-matrices of T. Going back to the Kruskal's criterion, in the symmetric case one has, just by symmetry, $Z_1 = Z_2 = Z_3 = Z$, and Z corresponds to the inverse image in the Veronese map of the set of points P_i defined by the expression of T. Recent refinements of the criterion for symmetric tensors follow by considering, together with the Kruskal rank of Z, also the Kruskal rank and the dimensions of the linear spans of Veronese images of Z.

A weakness of Kruskal's criterion resides in the fact that it only considers the geometry of the sets Z_1, Z_2, Z_3, in its original formulation, or the unique set Z, in the symmetric version. This means that in order to establish the identifiability of a point $[T]$ in the linear span of the Veronese image of Z, only the geometry of Z is involved, and not the position of $[T]$ in the span. In practice this means that Kruskal's criterion applies only if identifiability holds uniformly for all tensors in the linear span, excluding at most those spanned by the image of proper subsets of Z. We know now that this happens only when the cardinality r is small, and far from the value of the generic rank. In [1] it is illustrated how, already in the case of $Sym^9(\mathbb{C}^3)$ and generic sets $Z \subset \mathbb{P}(\mathbb{C}^3) = \mathbb{P}^2$ of cardinality 18, the generic point in the span Λ of the image of Z is identifiable, but Λ also contains points which lie in the span Λ' of the image a second set W of cardinality 18, disjoint from Z. Moreover Λ also contains points corresponding to tensors of rank 17, thus contained in the span of 17 points of the Veronese variety, which are disjoint from the image of Z. In order to detect the identifiability when r grows criteria like Kruskal's are thus not sufficient, and it is necessary to employ more sophisticated geometric tools. One tool involves the homogeneous ideal $\mathcal{I}_Z$ of $Z \subset \mathbb{P}^n$, i.e. the ideal generated by forms in $n+1$ variables that vanish at the points of Z. It is a standard fact in projective geometry that when $n = 2$ then $\mathcal{I}_Z$ is generated by the maximal minors of a matrix of forms, the 'Hilbert-Burch' matrix of Z. A procedure that uses the Hilbert-Burch matrix of Z to detect the identifiability points in the span of the Veronese image of Z is explained in detail in [1]. Since the Hilbert-Burch matrix can be effectively computed from the coordinates of the points of Z, by symbolic algebra computer packages, then we have a way of computing alternative expressions of T as a sum of powers, when they exist.



$\mathbb{P}(Sym^9(\mathbb{C}^3))$ and its secants

The classical works of J. J. Sylvester contained a reasonably complete description of the computation of the symmetric rank of tensors in $Sym^k(\mathbb{C}^2)$, corresponding to forms in two variables. The procedure illustrated above for the study of forms in three variables is considerably more complicated and requires more sophisticated projective tools, such as the Hilbert-Burch matrix. Indeed in three variables the global scenario involves so many possibilities that only an illustration of procedures that determine decompositions of forms as a sum of powers is possible. Nevertheless, I dare say that we are now at a level of understanding of the structure of forms in three variables similar to the level at which Sylvester, 140 years ago, handled forms in two variables . For forms in more variables our knowledge of the geometry of finite sets in higher dimensional projective spaces is not yet sufficient to provide an answer to the main questions on the structure of forms as symmetric tensors, except for a few initial cases. The interplay between projective geometry and tensor theory suggests, in both fields, a long series of related open problems.

What is next

I describe briefly, in this last section, some recent studies on the projective geometry of tensor spaces.

Until now I concentrated mainly on answers to question Q2 above. Answers to Q1 and Q3, i.e. strategies to compute a minimal expression of T as a sum of products, and thus the rank of T, are more developed in the symmetric case. Given a symmetric tensor and the associated form F, say of degree d in $n+1$ variables $x_0,\dots,x_n$, one can construct the 'apolar' ideal of F as follows. Consider an auxiliary polynomial ring $S=\mathbb{C}[y_0,\dots,y_n]$, whose variables can be interpreted as partial derivatives acting on the x_i's with the usual rules $y_i(x_j)=$ the Knonecker symbol δ_{ij}. Then the apolar ideal A_F of F is defined by polynomials $p(y)\in S$ that, in the interpretation as derivatives, kill F. See [10] for more details. The apolar ideal is classically known to be connected with sums of powers that define F. Namely $Z\subset\mathbb{P}^n$ is a finite set whose image in the d-th Veronese map spans $[F]$ if and only if the homogeneous ideal of F, in the new variables y_i, is a subset of A_F. An algorithm introduced in [3] and with several implementations (e.g. by A. Bernardi and D. Taufer, 2018) uses the multiplication map inside the apolar ideal to construct explicitly homogeneous ideals of (often minimal) sets of points inside A_F. It is indeed easy to construct ideals in A_F corresponding to finite sets Z of points. The difficult part concerns the minimality of the cardinality of Z. The combined use of the algorithms with projective methods that construct alternative, possibly minimal sets, such as the ones described at the end of the previous section, has not been explored so far.

In the construction of secant and abstract secant varieties I stressed that tensors of a given rank only fill an open dense subset of the corresponding secant variety. The closure contains limits of expressions in which two points can collapse together, or more generally expressions with linearly dependent summands. The limit cases no longer provide expressions of T as a sum of products. Yet, the study of the structure of limits can be important to prove the minimality of a given expression, or to produce new, minimal expressions. Recently W. Buczynska and J. Buczynsky (2021, see [4]) proposed to consider non-saturated ideals as tools to understand limit cases. Non-saturated homogeneous ideals do not correspond directly to geometric sets, yet their algebraic structure can be helpful for the study of specific tensors. For instance, understanding the behavior of non-saturated limit ideals which are stable under the action of large automorphism groups is likely to produce new insights in the investigation of the complexity of some multilinear processes, such as matrix multiplications. The investigation of limits in connection with the projective techniques illustrated above is still unexplored.

In the description of projective tools for the analysis of tensors I focused mainly on the symmetric case. The reason is that the theory of finite subsets of a Segre variety, which means finite subsets of products of projective spaces, is much less developed than the theory of finite sets in a Veronese variety, i.e. in a single projective space. For instance, a basic tool for the geometry of finite sets $Z\subset\mathbb{P}^n$ is the 'Hilbert function' H_Z, which associates to each positive integer i the (affine) dimension of the linear span of the image of Z in the Veronese map of degree i. A large literature describes properties of the Hilbert functions of finite sets in $\mathbb{P}^n$, even if the situation is well understood only for $n=2$. The Hilbert function of ideals contained in the apolar ideal of a form F provides many results on the rank of symmetric tensors. Similar functions are available for subsets of Segre varieties, but very little is known about their properties. A systematic study of finite subsets in products of projective spaces will open new perspectives, not only for applications in tensor theory, but also for the general development of projective geometry.

Finally, let me stress that there are several possible alternative notions for the rank of tensors, all with deep meaning in tensor theory. For instance, in the case of symmetric tensors I always used the 'symmetric rank', which is determined by expressions as sums of symmetric tensors of rank 1 (i.e. powers). Of course symmetric tensors can be also expressed as sums of non-symmetric products, and consequently they have a rank when considered as general tensors. The problem of whether the symmetric rank and the 'general' rank of symmetric tensors coincide was raised years ago, and recently solved by Y. Shitov, who constructed a symmetric tensor T for which the two notions of rank disagree (2018). Shitov's example is quite particular, for it belongs to (the limit in) a secant variety $\sigma_t(\mathbb{X})$ where t is much smaller than the rank of T. A deeper geometric description of the relation between general rank and symmetric rank of symmetric tensors, and conditions under which they coincide, is still under investigation.

I have concentrated on the state of the art for tensors defined over the complex field. Of course in many applications attention is focused on tensors defined over $\mathbb{R}$. There are easy examples of real tensors for which all minimal expressions as a sum of products involve non-real summands. These tensors have a 'real' rank which is bigger than the complex rank. The theory of the real rank of real tensors is another wide sector in which projective techniques (from real projective geometry) produce continuous advances.

There is a world of problems in projective geometry which are waiting for researchers eager to attack them, and to apply the solutions to increase our knowledge of tensors.

FURTHER READING

[1] E. Angelini, L. Chiantini, Minimality and uniqueness for decompositions of specific ternary forms, Math. of Comput. 91 (2022) 973-1006.
[2] J. Alexander, A. Hirschowitz, Polynomial interpolation in several variables, J. Algebraic Geom. 4 (1995) 201-222.
[3] J. Brachat, P. Comon, B. Mourrain, E. Tsigaridas, Symmetric tensor decomposition, Lin. Alg. Applic. 433 (2010) 1851–1872.
[4] W. Buczynska, J. Buczynsky, Apolarity, border rank, and multigraded Hilbert schemes, Duke Math. J. 170 (2021) 3659-3702.
[5] M. V. Catalisano, A. V. Geramita, A. Gimigliano, Secant varieties of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (n-times) are not defective for $n \geq 5$, J. Algebraic Geom. 20 (2011) 295-327.
[6] L. Chiantini, C. Ciliberto, On the concept of k-secant order of a variety, J. London Math. Soc. 73 (2006) 436-454.
[7] L. Chiantini, G. Ottaviani, N. Vannieuwenhoven, On generic identifiability of symmetric tensors of subgeneric rank, Trans. Amer. Math. Soc. 369 (2017) 4021-4042.
[8] I. Domanov, L. de Lathauwer, On the uniqueness of the canonical polyadic decomposition of third-order tensors - part II: Uniqueness of the overall decomposition, SIAM J. Matrix Anal. Appl. 34 (2013) 876-903.
[9] F. Galuppi, M. Mella, Identifiability of homogeneous polynomials and Cremona transformations, J. Reine Angew. Math. 757 (2019) 279-308.
[10] A. Iarrobino, V. Kanev, Power Sums, Gorenstein Algebras, and Determinantal Loci, Lecture Notes in Mathematics 1721, Springer, 1999.
[11] J. M. Landsberg, Tensors: geometry and Applications, Graduate Studies in Mathematics 128, American Mathematical Society Press, 2012.

Luca Chiantini
Luca is a professor in geometry at the University of Siena, Italy. His research interests are in algebraic geometry, algebraic statistics, commutative and multilinear algebra.

Gudea's Measuring Rod

LLEONARD RUBIO Y DEGRASSI

Gudea was a ruler of the state of Lagash who lived in 3rd millennium BC. One of the statues of Gudea depicts the prince with a temple plan, a stylus and a measuring rod on his lap. In this article we explain how this measuring rod reflects part of the mathematical knowledge and conventions of the Sumerians.

Ruler and base 60

Have you ever thought why in a ruler 1 unit is divided into 10 equal parts? Put simply, the answer is that we use base 10. One could then ask: how would a ruler look if we were using another base, for example base 60?

For some people, this question might seem too abstract and not very practical. However, such a ruler exists and it is 4000 years old. In order to understand how this ruler works, we need to go back in time 4000 years and travel to the area that is present-day Iraq.

For me this fascinating journey began during a conversation with Sebastien Rey, curator of Ancient Mesopotamia at The British Museum and Director of the Girsu Project. One evening he told me that he was working on establishing a deeper connection between his recent excavations at the temple site in Girsu and a statue of Gudea, called the Gudea temple plan; he held on his knee a plan of this temple.

Gudea's measuring rod

Gudea's plan was carved at the end of the 3rd millennium BC and it is one of the earliest-known architectural plans in history (Figure 1). Together with the plan there is a stylus (on the left hand side) and a measuring rod (at the bottom). Although the measuring rod was not designed exactly as modern rulers, we think that the design of this ruler represents the base used by the Sumerians, in a similar way that the modern rulers represent the base used in modern times. It was pointed out by Julien Chanteau and Sebastien Rey [3] [5] that if we read the ruler from left to right (in Figure 1), then the metric system is based on one unit (SU) also called the 'sacred unit' repeated six times, each time subdivided into increasing fractions:

$$\frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1$$

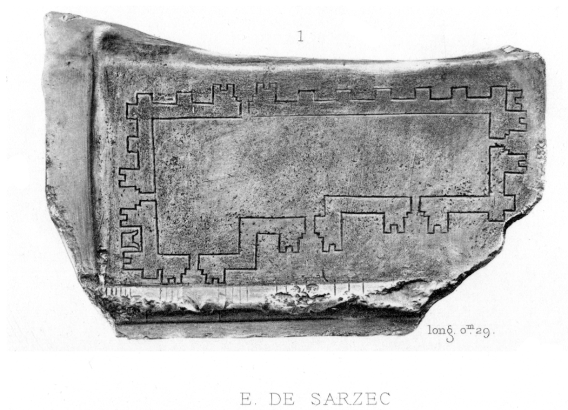


Figure 1: Reproduced from Sarzec and Heuzey, Découvertes en Chaldée 1912, Pl. 15

One natural question to ask is "why do Sumerians stop dividing the unit at 6 parts?" This was precisely the question that I had during my conversation with Sebastien Rey regarding Gudea's measuring rod.

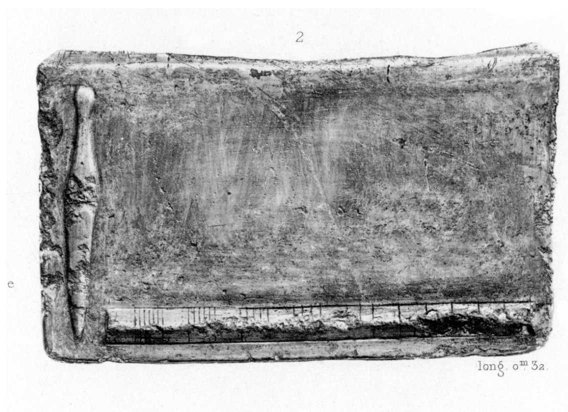


Figure 2: Reproduced from Sarzec and Heuzey 1912, Pl. 15

One could think that maybe the precision required was not enough for carving further divisions. However, we excluded this possibility thanks to another ruler, in which there are further marks on the bottom left corner (in Figures 2 and 3) which represent $\frac{1}{12}$ and $\frac{1}{18}$.

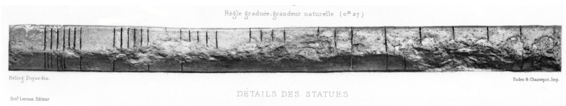


Figure 3: detail of Figure 2

In order to answer this question we need to understand the mathematical tools that Sumerians had.

Sumerian-Babylonian sexagesimal system

The notion of a number in mathematics has changed and extended throughout history. Four thousand years ago, the notion scribes had of a number was a finite sequence of sexagesimal digits $0, 1, \ldots, 59$ where both the first and last digits were non-null [2]. The digits were written in terms of multiples of 10 (Figure 4). For the null digit, scribes used to leave a blank. To separate the digits of a number we will use a colon. If not otherwise stated, this is the notion of a number that we will use. This is called the *sexagesimal place value system*, and it is usually abbreviated as SPVS.

Note that the loss of surrounding null digits implies that the numbers were only determined up to a multiple of 60. For example, 5 could mean 5 or $\frac{5}{60}$ or $\frac{5}{3600}$ and so on. This ambiguity was usually resolved by the context.

One of the advantages of using a sexagesimal system is that 60 has twelve factors (compare it with 10 which has only four factors):

$$1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$$

Having many factors is very useful. In order to understand this in detail let us first introduce some definitions. A number p, written in the SPVS, is called *regular* if there exists another number q such that $pq = 60 = 1$ (remember that in the SPVS 60 also represents 1). We call q the *reciprocal* of p. Any number which is not regular is *irregular*.

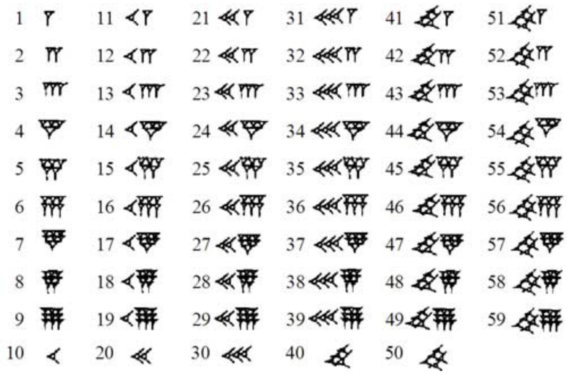


Figure 4: Babylonian cuneiform numerals, source: Wikipedia

For example, the number 3 is regular since $3 \cdot 20$ is equal to 1 in the SPVS. Another example of a regular number is 12 since there exists a number, in this case 5, having a finite sequence of sexagesimal digits, such that $12 \cdot 5 = 60$. An example of an irregular number is 7.

Any regular number is of the form $2^a \cdot 3^b \cdot 5^c$ where $a, b, c \in \mathbb{Z}$. In order to remember the regular numbers and their reciprocals, in the Old Babylonian period (1900-1600 BCE) the standard table of reciprocals were an essential tool for the scribes. The reader can find below the table of reciprocals of the first regular numbers:

1	1
2	30
3	20
4	15
5	12
6	10

A natural question emerges: why were regular numbers important for ancient Sumerians and Babylonians? One of the main reasons is because, as far as we know, Sumerians and Babylonians used only the number system that we have described. In modern mathematics the situation is pretty different since aside from the decimal system we also use fractions. With this viewpoint in mind, having more regular numbers is extremely helpful.

Indeed, in the decimal system the regular numbers are only of the form $2^a \cdot 5^b$, where $a, b \in \mathbb{Z}$. For example, we cannot write $\frac{1}{3}$ in the decimal system using a finite number of digits. However, this is not a problem in the sexagesimal (place value) system because $\frac{1}{3}$ is equal to 20. Therefore, the ancient Sumerians and Babylonians could represent many more numbers by a finite number of digits without using the notion of rational numbers.

Localization

The algebraic notion that generalises decimal and sexagesimal systems and rational numbers is called *localisation*. Roughly speaking, localisation allows us to invert a (mutiplicative) set of elements, that is, it allows us to introduce formally the notion of "denominators".

Let R be a commutative ring. In fact, in order to simplify the exposition and to allow us to introduce more familiar concepts, we will assume that $R = \mathbb{Z}$ the set of integers. It is worth noting, however, that ancient Sumarians did not have the notion of integers.

What is a commutative ring?

A ring is an abelian group R with a multiplication operation $(a, b) \mapsto ab$ and an identity element 1 satisfying for all $a, b, c \in R$:

- a(bc)=(ab)c
- a(b+c)=ab+ac
- (b+c)a=ba+ca
- 1a=a1=1

A ring R is commutative if in addition $ab = ba$ for all $a, b \in R$.

Before recalling the notion of localisation we need to introduce a definition. A *multiplicative set* S is a subset of R such that the multiplicative identity 1 is in S and the product xy of two elements $x, y \in S$ is in S. We will consider three multiplicative sets of $\mathbb{Z}$ as guidelines. The first is given by the powers of 10, that is, $S_1 = \{1, 10, 10^2, \dots\}$. The second by the powers of 60: $S_2 = \{1, 60, 60^2, \dots\}$ and the third one $S_3 = \mathbb{Z} \setminus \{0\}$. The reader can check that these are multiplicative sets.

We define an equivalence relation on $R \times S$ as follows: two pairs (r, s) and (r', s') are in relation if $rs' = r's$. We can think of this equivalence relation as a generalization of the condition of two fractions being equivalent: $\frac{r}{s} = \frac{r'}{s'}$ if and only if $rs' = r's$.

Equivalence relation on $R \times S$

Since the commutative ring $\mathbb{Z}$ does not have zero divisors, that is, elements x such that there exists a nonzero element y such that $xy = 0$, our definition gives an equivalence relation.

We denote the equivalence class of pairs $(r, s) \in R \times S$ by $\frac{r}{s}$. The set of all equivalence classes is denoted

$$S^{-1}R = \{\frac{r}{s} \mid r \in R,\ s \in S\}$$

and is called the *localization* of a ring R at a multiplicative set S. There is a natural ring map, sometimes called the localization map,

$$\varphi : R \to S^{-1}R$$

which sends $r \in R$ to $\frac{r}{1}$. If S contains no zero divisors then φ is injective, so we can see the elements of R as elements of $S^{-1}R$. Note that $S^{-1}R$ has a commutative ring structure where the sum $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$ and the product $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$. The process of localisation allow us adjoin the inverses of all elements of a set S. Indeed, if $s \in S$, then the inverse of $\frac{s}{1}$ in $S^{-1}R$ is $\frac{1}{s}$.

Let us look at some examples: if we localise with respect to S_1, then

$$S_1^{-1}R = \{\frac{r}{s} \mid r \in \mathbb{Z},\ s = 10^i,\ i \in \mathbb{N}\},$$

that is,we obtain the decimal system. We claim that the invertible elements in $S_1^{-1}R$ are all elements $2^a \cdot 5^b$, where $a, b \in \mathbb{Z}$. It is enough to show this for elements of the form 2^a since for the rest of the elements the proof is analogous. We have seen that all (equivalence classes of) elements of the form $(s, 1)$ for $s \in S_1$ are invertible. Take an element of the form 2^a, that is $(2^a, 1)$ for some $a \in \mathbb{Z}$. Since the inverse of $(10^a, 1)$ is $(1, 10^a)$, we have the following equalities: $(10^a, 1) \cdot (1, 10^a) = (10^a, 10^a) = (1, 1)$ in $S_1^{-1}R$. Then $(5^a, 10^a)$ is the inverse of $(2^a, 1)$. The statement follows.

Similarly, if we localise with respect to S_2 we obtain the sexagesimal system. The invertible elements are all elements $2^a \cdot 3^b \cdot 5^c$, where $a, b, c \in \mathbb{Z}$. If we localise with respect to S_3 we obtain the field of rational numbers $\mathbb{Q}$. Recall that in this case all non-zero elements are invertible.

In particular, the element $(3, 1) = \frac{3}{1}$ in $S_1^{-1}R$ does not have an inverse while the inverse of $(3, 1) =$

$\frac{3}{1}$ in $S_2^{-1}R$ is $\frac{20}{60}$. Consequently we can say that scribes used to localise with respect to S_2, while if we consider a decimal system, we localise with respect to S_1. In order to invert all elements we need the rational numbers, that is, we need to localise with respect to S_3.

So how does this construction allow us to understand Gudea's measuring rod better? If we represent these fractions in the sexagesimal place value system

$$10, 12, 15, 20, 30, 60$$

and we look at the table of reciprocals we might get a plausible answer: all these numbers are regular. In fact they are the first regular numbers (also $\frac{1}{12}$ and $\frac{1}{18}$ are regular numbers). This also explains why they did not divide in seven parts: 7 is an irregular number.

In addition, this ruler can be thought a partial geometrical table of reciprocals. More precisely, given a segment of length 60, we divide it into segments of length $10, 12, 15, 20, 30, 60$, respectively. If we multiply the number of times in which the segment has been divided by the length of each segment we obtain 60. Hence, the reciprocal of a number is encoded geometrically. We say that this is a partial geometrical table of reciprocals since the standard table of reciprocals used to contain more numbers and the inverses were not encoded geometrically. In order to validate our assumptions, we have measured the different gaps between the marks of the measuring rod [4].

To draw an analogy, we could think that a ruler in a decimal place value system constructed using this idea would consist only of two segments: one representing the unit and another divided into two parts.

In addition, we have observed that some of the measurements of the temple can be expressed in relation to the measuring rod. Specifically, the entrances of the temple are either of width 15 or 20. Another example is given by the heights of the entrance towers which vary from 30 to 50, in Figure 6.

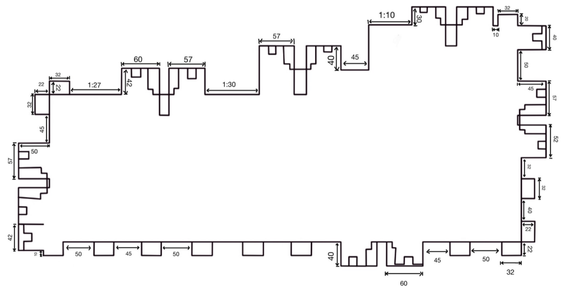


Figure 6

More exciting discoveries have been made by Sebastien Rey in Tello (the modern Arabic name for the ancient Sumerian city of Girsu) who has used the principles of the 'sacred unit' to pinpoint where a gate of the hidden temple should be and opened a trench [3]. After three weeks of excavation he found the foundation of the gate.

Although there are still many open questions about this temple and its plan, it is interesting to note how the tools used in ancient civilizations were based on the mathematical notions that they had developed.

Acknowledgements

The author was supported by the Iraq Scheme and by the Girsu Project.

FURTHER READING

[1] D. Eisenbud, Commutative Algebra with a View Toward Algebraic Geometry, Graduate Texts in Mathematics.
[2] D. F. Mansfield, Plimpton 322: A Study of Rectangles, Foundations of Science (2021).
[3] S. Rey, Cracking Gudea's Code (2019), British Museum Magazine, (2019).
[4] S. Rey, L. Rubio y Degrassi, Gudea's temple plan, in preparation.
[5] Whisky galore solves ancient mystery, The Times, March 21, (2019).
[6] Sarzec, E. de and Heuzey, L. (avec le concours de A. Amiaud et de F. Thureau-Dangin) Découvertes en Chaldée, Paris: Ernest Leroux, (1912).

Lleonard Rubio y Degrassi
Lleonard is a postdoc in mathematics at Uppsala University. His research interests are in homological algebra and representation theory.

Notes of a Numerical Analyst

Which is Smaller, $O(n^2)$ or $O(n^3)$?

NICK TREFETHEN FRS

An old dream is the "Fast Matrix Inverse", which would invert an $n \times n$ matrix in essentially $O(n^2)$ operations — $O(n^2 \log n)$, perhaps. Such a discovery would revolutionise computational science, as the FFT revolutionised signal processing with its $O(n \log n)$ operation count for an n-point discrete Fourier transform.

But despite the importance of the problem, nobody has ever found the FMI, nor proved that it cannot exist. Mostly we use the classical $O(n^3)$ algorithms. There are theoretical alternatives needing just $O(n^{2.37})$, but the constants are enormous.

I was discussing these matters with a colleague the other day who startled me by saying, "But computers already achieve $O(n^2)$! Just give it a try on your machine!"

I did that, and the result is shown in Figure 1. Sure enough, for small n, the shape looks like $O(n^2)$. A user working with $n < 1000$ might think that the FMI already exists and is running on their laptop. On the other hand for $n \gg 1000$ we see equally cleanly $O(n^3)$, as we learned in our numerical analysis courses.

One could discuss why these results look the way they do, but my interest is in the more basic question, what do they *mean?* Would it be fair to say "Yes, it's $O(n^3)$ in theory, but the bad running time doesn't kick in until n is quite large"?

For there is a paradox here: the computation would obviously be faster if there were no $O(n^2)$ component at all and the $O(n^3)$ kicked in right from the start. Or how about this: if the running times were longer by $2 \cdot 10^{-5}\, n$, the complexity would look beautifully like $O(n)$ for $n < 1000$, but of course that would not be a better algorithm.

Analogously, I've seen people assert that although exponential convergence is provably impossible for a certain problem, they've got a method that "converges exponentially down to any specified accuracy $\varepsilon > 0$". You can depend upon it, the exponential initial transients of such a method lie above a subexponential envelope.

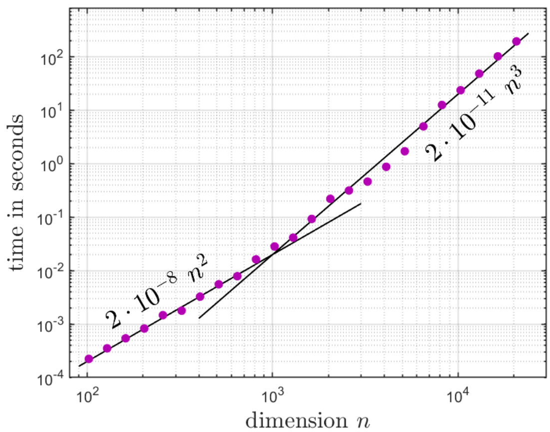


Figure 1. Inverting an $n \times n$ matrix on my laptop.

The disturbingly plausible idea that $O(n^2) + O(n^3)$ might be somehow faster than $O(n^3)$ alone reminds me of a moment in *Through the Looking-Glass.*

"It's a poor sort of memory that only works backwards," the Queen remarked.

"What sort of things do *you* remember best?" Alice ventured to ask.

"Oh, things that happened the week after next," the Queen replied in a careless tone. "For instance, now, . . . there's the King's Messenger. He's in prison now, being punished; and the trial doesn't even begin till next Wednesday: and of course the crime comes last of all."

"Suppose he never commits the crime?" said Alice.

"That would be all the better, wouldn't it?" the Queen said.

Nick Trefethen
After 26 years at Oxford, Trefethen has moved to Harvard University, where he is Professor of Applied Mathematics in Residence.

Mathematics News Flash

Jonathan Fraser reports on some recent breakthroughs in mathematics.

The Eremenko-Lyubich constant

AUTHORS: Lasse Rempe
ACCESS: https://arxiv.org/abs/2105.09053

In an influential paper from 1992, Eremenko and Lyubich introduced the (now eponymous) class of transcendental entire functions whose set of critical and asymptotic values is bounded. In particular, for a given f in this class, we may choose $R > 0$ large enough such that $|f(0)| \leq R$ and such that the moduli of all critical and asymptotic values of f are also bounded above by R. Then f is expanding at points where the modulus is large. More precisely, Eremenko and Lyubich proved that

$$|f'(z)| \geq \frac{(\ln|f(z)| - \ln R)|f(z)|}{4\pi|z|}.$$

This paper, published in *Bulletin of the London Mathematical Society* in 2023, proves that 4π may be replaced by 2 in the above and that this is optimal. The readers might like to test the optimality of 2 for themselves by considering $f(z) = \cos\sqrt{z}$.

Haar Null Closed and Convex Sets in Separable Banach Spaces

AUTHORS: Davide Ravasini
ACCESS: https://arxiv.org/abs/2110.05250

A locally compact (Hausdorff) topological group admits a *Haar measure*, that is, a translation invariant "volume" defined on the Borel subsets of the group. This measure is unique up to normalisation. For example, the Haar measure on $(\mathbb{R},+)$ giving mass 1 to $[0,1]$ is Lebesgue measure (restricted to the Borel sets). Motivated by the lack of a Haar measure in the non-locally compact case, Christensen defined the notion of *Haar null* to extend the idea of "Haar measure zero" to a more general setting. In a similar fashion, Darji introduced the concept of *Haar meagre*, which is a categorical analogue of "being small".

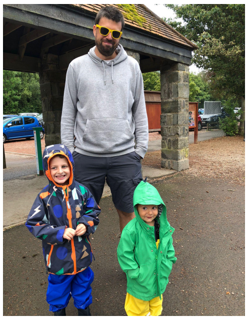
Although equally natural, the notions of Haar null and Haar meagre are very different. For example, the set of normal numbers in $\mathbb{R}$ form a Haar meagre set, but the complement of this set is Haar null. (The readers may enjoy proving this for themselves, after looking up the necessary definitions!) Given this difference, there is interest in determining conditions under which the notions agree. This paper, published in *Bulletin of the London Mathematical Society* in 2023, proves that a closed, convex subset of a separable Banach space is Haar null if and only if it is Haar meagre.

An invariant property of Mahler measure

AUTHORS: Matilde Lalín, Siva Sankar Nair
ACCESS: https://arxiv.org/abs/2211.00125

The *Mahler measure* of a (multivariate) polynomial over $\mathbb{C}$ is a subtle quantity which in a certain sense measures the complexity of the polynomial. The Mahler measure has applications to various problems in number theory.

There have been several papers connecting the Mahler measure of polynomials of particular interest to expressions involving Dirichlet L-functions and the Riemann zeta function ζ. For example, Condon studied the polynomial $x + 1 + (x - 1)(y + z)$ and proved that it has logarithmic Mahler measure given by $\frac{28}{5\pi^2}\zeta(3)$. (I am less inclined to suggest this as an exercise for the reader.) This paper, published in *Bulletin of the London Mathematical Society* in 2023, proves that the Mahler measure is preserved under a certain change of variables. This interesting observation came from first observing that several polynomials which had all been shown (numerically) to have a particular Mahler measure could all be obtained from Condon's polynomial by clever changes of variables.



Jonathan Fraser is a pure mathematician working at the University of St Andrews. He is interested in fractal geometry, dynamical systems, and analysis and is ready for all weather conditions.

Microthesis: An Information-Theoretic Model of T Cell Activation

JOSEPH R. EGAN

75 years ago Claude Shannon published his ground-breaking paper [1] in which he laid the foundation for the digital age. Here, I apply Shannon's theory to the communication between immune cells and harmful cells, and propose that the magnitude of the immune response depends of the entropy rate of this interaction.

T cells are a critical component of the adaptive immune system because they are responsible for seeking out and ultimately destroying harmful (e.g., cancerous or virus-infected) cells. Receptors located on the T cell surface can bind to ligands located on the surface of a harmful cell. This interaction can generate a signalling cascade within the T cell resulting in its own activation. There is increasing evidence that T cell activation can be induced by very low numbers of receptors and ligands. This means that their binding dynamics are inherently stochastic; however, the effect of these fluctuations on T cell activation is unclear. Here, I propose that T cell activation depends on a stochastic property known as the entropy rate, which is the average rate at which receptor-ligand interactions generate information (in bits per second).

A simple model of receptor-ligand binding is given by the reversible heterodimerisation reaction:

$$L + R \underset{k_{\text{off}}}{\overset{k_{\text{on}}}{\rightleftharpoons}} B\,, \tag{1}$$

where L denotes a ligand, R denotes the T cell receptor, B denotes the bound complex, k_{off} is the rate of unbinding, k_{on}/v is the rate of binding and v is the 2-dimensional (2D) contact area in which the biochemical reactions take place. The 2D dissociation constant, K_{d}, which is a measure of the strength of interaction between ligand and receptor, is given by:

$$K_{\text{d}} = \frac{v k_{\text{off}}}{k_{\text{on}}}. \tag{2}$$

The entropy rate, and related variance rate, are derived via a stochastic treatment of Eq. 1 (see "Key equations of the entropy rate").

Key equations of the entropy rate

Let B_{max} and U_{max} denote the smaller and larger, respectively, of the total number of ligands and receptors. The stationary probability distribution of the bound complex number, $p(B)$, is given by:

$$p(B) = \frac{a(B)}{\sum_{i=0}^{B_{\text{max}}} a(i)}, \tag{3}$$

where

$$a(B) = \binom{B_{\text{max}}}{B}\binom{U_{\text{max}}}{B} {K_{\text{d}}}^{-B} B!, \tag{4}$$

and where K_{d} is given by Eq. 2. The mean, $\langle B\rangle$, variance, $\text{Var}(B)$, and Shannon entropy, $H(B)$, of $p(B)$ are given by:

$$\langle B\rangle = \sum_{i=0}^{B_{\text{max}}} i\,p(i), \tag{5}$$

$$\text{Var}(B) = \sum_{i=0}^{B_{\text{max}}} i^2 p(i) - \langle B\rangle^2, \tag{6}$$

$$H(B) = -\sum_{i=0}^{B_{\text{max}}} p(i)\log_2(p(i)). \tag{7}$$

Note that the Shannon entropy is the average amount of information generated (in bits) per reaction. In contrast, the entropy rate, $H'(B)$, is the average rate at which information is generated (in bits/second) given by:

$$H'(B) = 2k_{\text{off}}\langle B\rangle H(B), \tag{8}$$

A closely related property of the entropy rate is the 'variance rate', $\text{Var}'(B)$, given by:

$$\text{Var}'(B) = 2k_{\text{off}}\langle B\rangle\,\text{Var}(B). \tag{9}$$

Figure 1 shows stochastic simulations of the bound complex copy-number (blue lines) of Eq. 1. Each row of panels has the same mean number of bound complexes (red dashed lines) and Shannon entropy (a property similar to the variance, the square root of which is shown by the red dotted lines relative to the dashed). This is because the total number of ligands and receptors is fixed and the 2D dissociation constant is the same for each row. However, while the mean monotonically increases in each column as the binding rate increases, both the Shannon entropy and entropy rate initially increase before reaching a maximum and then decreasing. Moreover, unlike the Shannon entropy, the entropy rate is an order of magnitude higher in the right hand column than the left because the unbinding rate is an order of magnitude higher. In my thesis I show that the entropy rate is consistent with experimental data of cell outputs related to T cell activation.

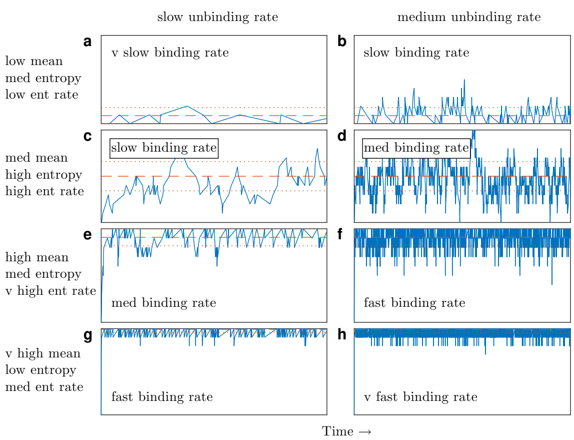


Figure 1. Fluctuations in receptor-ligand interactions generate information.

Figure 2 shows three known mechanisms by which a T cell transmits a signal to its interior. Specifically, (1) the solid black arrows represent a ligand serially engaging with multiple receptors (2) The green receptor represents a change in receptor shape upon ligand binding. The dashed black arrow represents the receptor reverting back to its original shape after unbinding. (3) The dotted black arrow represents receptor aggregation following ligand binding. The combination of these mechanisms generates a signal, S within the T cell. In my thesis, I show that the mean signalling rate is approximately equal to half the so-called 'variance rate', which is an analytically convenient proxy for the entropy rate (see "Key equations of the entropy rate").

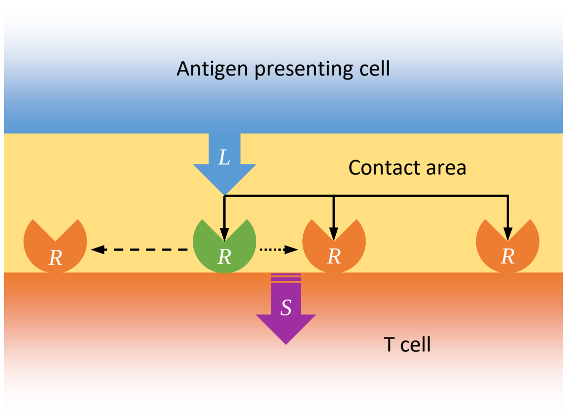


Figure 2. Schematic of the three modelled mechanisms.

In summary, stochasticity in receptor-ligand binding dynamics generates a 'message'. The entropy rate is the average information content per second of this message. From an immunological perspective, the entropy rate can be interpreted as the average rate at which antigen-specific information is imparted to the T cell. Based on this reasoning, and the supportive experimental data, it is proposed that T cell activation is a function of the entropy rate. Optimising the entropy rate via manipulation of its parameters could potentially enhance existing immunotherapies designed to treat cancer and other diseases.

This microthesis resulted from an Early Career Fellowship grant from the LMS. A related Fellowship Report is available at bit.ly/newsletter-500. The research summarised in this microthesis is available as a full thesis at eprints.soton.ac.uk/452391 and as a related paper at royalsocietypublishing.org/doi/10.1098/rsif.2021.0589.

FURTHER READING

[1] C.E. Shannon, A Mathematical Theory of Communication, Bell System Technical Journal (1948)

Joseph R. Egan

Joe is a post-doctoral researcher in biochemical engineering at University College London. His research interests include mathematical modelling of cell therapies and his published work is available **here**. Joe lives in Winchester, Hampshire, where he enjoys running and coaching his sons' football teams.

Death Notices

We regret to announce the following deaths:

- Professor David A. Burguess, formerly of the University of Nottingham, who was elected an LMS member on 16 April 1959, died on 27 July 2023, aged 88.
- Professor Sean J. Tobin, formerly of University College, Galway, who was elected an LMS member on 18 October 1985, died on 11 July 2023, aged 93.
- Professor Gordon Elliott 'Tim' Wall, formerly of the University of Sydney, who was elected an LMS member on 12 May 1955, died on 9 July 2023, aged 98.

Society Meeting

Northern Regional Meeting & Workshop 2023

4–6 September 2023, King's Manor, University of York

Website: lms.ac.uk/events/meeting/lms-northern-regional-meeting-2023

Website: sites.google.com/view/lms-york-2023/home

The meeting forms part of the Northern Regional Workshop on Modular Lie Theory from 5-6 September. The meeting will open with Society Business, during which LMS members will have the opportunity to sign the Members' Book. This will be followed by a Lecture, to be given by Anne Schilling (UC Davis) on *The mystery of plethym*, and then lectures will be given by Lewis Topley (Bath) on *What the W?* and Beth Romano (KCL) on *Graded Lie algebras and applications to number theory*.

A reception at King's Manor will be held after the meeting, and a Society Dinner will be held after the meeting. If you would like to attend the dinner, please contact the organisers: Michael Bate (michael.bate@york.ac.uk), Chris Bowman (chris.bowman-scargill@york.ac.uk), Harry Geranios (haralampos.geranios@york.ac.uk), Amit Hazi (amit.hazi@york.ac.uk). Funds are available for partial support to attend. Requests with an estimate of expenses should be addressed to the organisers.

Analysis of Fluid Equations at Bath

Location: University of Bath
Date: 8 September 2023
Website: sites.google.com/view/bath-lms-fluids

This workshop aims to bring together researchers in nonlinear partial differential equations arising from fluids, such as the Euler and Navier–Stokes equations. Early career researchers and PhD students are encouraged to attend and to communicate their research as a poster presentation. Registration is free but mandatory. The registration link can be found on the workshop website. Deadline for registration is 4 September 2023. The workshop is funded by the LMS Celebrating New Appointments scheme and the University of Bath.

Employers Forum — AI, Automation and Mathematics in Supporting Business Decisions

Location: Warwick University
Date: 20 September 2023
Website: tinyurl.com/4mkkj5m4

The Institute of Mathematics and its Applications is hosting this Employer's Forum. This will be an opportunity for businesses, including SMEs, to explore how AI and mathematics can impact business in the widest sense. There will be a keynote speaker and four further short presentations on topics ranging from automation of computer code to probabilistic modelling for decision making under uncertainty. There will be poster presentations on a range of related topics and networking.

Operator Algebras and K-theory of Dynamical Systems

Location: Newcastle University
Date: 28 September 2023
Website: tinyurl.com/oaktds23

A one-day workshop on the interplay between topological dynamical systems, operator algebras and algebraic topology. The event will feature an introductory talk to this area of research by Christian Bönicke (Newcastle), as well as invited research talks by Jacqui Ramagge (Durham) and Xin Li (Glasgow). The workshop is open to everyone, and PhD students and postdocs are particularly encouraged to attend. This meeting is partly supported by an LMS Celebrating New Appointments grant.

LMS/IMA Joint Meeting 2023: The Mathematical Foundations of Artificial Intelligence

Location: London (De Morgan House)
Date: 13 October 2023
Website: tinyurl.com/joint2023

The speakers at this event will be: Gitta Kutyniok (LMU München), *Reliable AI: Successes, Challenges, and Limitations*; David Saad (Aston University), *The Space of Functions Computed by Deep-learning Networks*; Petar Velickovic (DeepMind and University of Cambridge), *Capturing Computation with Algorithmic Alignment* plus Michael Bronstein (University of Oxford) and Lisa Kreusser (University of Bath).

Black Heroes of Mathematics 2023

Location: Online via Zoom
Date: 3-4 October 2023
Website: tinyurl.com/bhom2023

The vision of this conference is 'To celebrate the inspirational contributions of black role models to the field of Mathematics and Mathematics Education'. There will be a balance of technical talks by internationally renowned black speakers that include some detail of career paths and experience to provide a testimonial dimension. We plan to achieve a balance of career stage and gender. This event is free to attend and all are welcome.

LMS Society Meeting and Mary Cartwright Lecture

Location: ICMS, Edinburgh
Date: 19 October 2023
Website: tinyurl.com/mary-2023

The meeting will open with Society Business, during which LMS members will have the opportunity to sign the Members' Book. This will be followed by an Accompanying Lecture, to be given by Dan Margalit (Vanderbilt University); *Reconstruction problems in mathematics: from Euclid to Ivanov*. The Mary Cartwright Lecture 2023 will then be given by Tara Brendle (University of Glasgow); *Ivanov's metaconjecture: encoding symmetries of surfaces*.

Society Meeting

LMS AGM and Presidential Address 2023

17 November 2023, Mary Ward House, London and online

Website: lms.ac.uk/events/annual-general-meeting-2023

The meeting will open with Society Business, during which LMS members will have the opportunity to sign the Members' Book. This will be followed by the accompanying Lecture, to be given by Oscar Randal-Williams (Cambridge). The Presidential Address 2023 will then be given by Professor Ulrike Tillmann (INI-Cambridge).

A reception will be held after the meeting, which will be held at Mary Ward House, and the Annual Dinner will be held after the meeting. If you would like to attend the dinner, please indicate this on the registration form and send payment using the details which will be given in the acknowledgement email.

UNIVERSITY OF CAMBRIDGE

ADAMS PRIZE

Geometry

The University of Cambridge has announced the subject for one of its oldest and most prestigious prizes. The Adams Prize is named after the mathematician John Couch Adams and was endowed by members of St John's College. It commemorates Adams' role in the discovery of the planet Neptune, through calculation of the discrepancies in the orbit of Uranus.

The Chairman of the Adjudicators for the Adams Prize invites applications for the 2023-2024 prize which will be awarded this year for achievements in the field of Geometry.

The prize is open to any person who, on 31st October 2023, will hold an appointment in the UK, either in a university or in some other institution; and who is under 40 (in exceptional circumstances the Adjudicators may relax this age limit). The value of the prize is expected to be approximately £30,000, of which one third is awarded to the prize-winner on announcement of the prize, one third is provided to the prize-winner's institution (for research expenses of the prize-winner) and one third is awarded to the prize-winner on acceptance for publication in an internationally recognised journal of a substantial (normally at least 25 printed pages) original article, of which the prize-winner is an author, surveying a significant part of the winner's field.

Applications, comprising a CV, a list of publications, the body of work (published or unpublished) to be considered, and a brief non-technical summary of the most significant new results of this work (designed for mathematicians not working in the subject area) should be sent to the Secretary of the Adams Prize Adjudicators via email to adamsprize@maths.cam.ac.uk.

The deadline for receipt of applications is 31st October 2023

More information is available at www.maths.cam.ac.uk/adamsprize

Society Meetings and Events

September

4-6 LMS Northern Regional Meeting & Workshop, University of York

October

3-4 Black Heroes of Mathematics, online
13 Joint Meeting with the IMA: The Mathematical Foundations of Artificial Intelligence, De Morgan House, London
19 LMS Society Meeting and Mary Cartwright Lecture, ICMS Edinburgh

November

17 LMS AGM and Presidential Address, Mary Ward House, London

January 2024

17 LMS South West and South Wales Regional Meeting and Workshop, Bath

June 2024

28 LMS General Meeting, London

Calendar of Events

This calendar lists Society meetings and other mathematical events. Further information may be obtained from the appropriate LMS Newsletter whose number is given in brackets. A fuller list is given on the Society's website (www.lms.ac.uk/content/calendar). Please send updates and corrections to calendar@lms.ac.uk.

September

7 Mathematics in Defence and Security IMA Conference, Imperial College, London (506)
8 Analysis of Fluid Equations at Bath, University of Bath (508)
6-8 36th British Topology Meeting, University of Sheffield (507)
7-8 British Logic Colloquium 2023, University of Bristol (507)
7-8 Heilbronn Annual Conference, University of Bristol (507)
18-19 London Mathematical Biology Conference, University College London (507)
20 Employers Forum, AI, Automation and Mathematics in Supporting Business Decisions, University of Warwick University (508)
25-29 Clay Research Conference and Workshops, Mathematical Institute, Oxford (507)
28 Operator Algebras and K-theory of Dynamical Systems, Newcastle University (508)

December

12-14 Cryptography and Coding Conference, Royal Holloway, London (507)